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## Tractable forms of the bond pricing equation

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# Tractable Forms of the Bond Pricing Equation

*A thesis submitted in fulfilment of the  
requirements for the award of the degree of*

**Honours Master of Science (Mathematics)**

*from*

**The University of Wollongong**

*by*

**Gaurav Raina, B.Math. (Honours)**

**School of Mathematics and Applied Statistics**

**1999**

## **CERTIFICATION**

I, Gaurav Raina, declare that this thesis, submitted in fulfilment of the requirements for the award of Honours Master of Science (Mathematics), in the School of Mathematics and Applied Statistics, University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. The document has not been submitted for qualifications at any other academic institution.

Gaurav Raina

14 November 2001

*To Phil and Joanna for their endless support and encouragement*

# Acknowledgements

A lot of things in life seem incredible to most men  
who have not studied mathematics.

Archimedes

I write these acknowledgements two years earlier than anticipated but can assure you that my gratitude is not time dependent. If one is to take Archimedes seriously then a Mathematics department could aptly be called the 'Crown' of any University. It was not only my pleasure to spend time in the Mathematics department at the University of Wollongong but a privilege to be supervised by the two 'Jewels in its Crown' Professor Phil Broadbridge and Dr. Joanna Goard. Unfortunately my gratitude for their endless encouragement cannot be expressed within the span of this acknowledgement. Their endless motivation has been a key factor in making this thesis see the light of day.

I also take this opportunity to thank all the members of the School of Mathematics and Applied Statistics for all their help and guidance and although I have headed off towards 'supposedly' greener pastures I must admit that it is not without a heavy heart and memories to cherish a life time.

# Abstract

A key area of study in the world of financial derivatives is the modelling of the short-term interest rate with a view to finding theoretically fair prices for financial instruments. We consider a second order linear partial differential equation of parabolic type which has the spot-rate (otherwise known as the *short-term* interest rate) and time as independent variables, and which can be used to model various financial instruments such as fixed-income products. In this thesis we have concentrated on finding analytic solutions to this equation for pricing simple bonds and hence refer to this equation as the Bond Pricing Equation (BPE). The non-constant coefficients of this equation originate from the drift coefficients and variable volatility in the underlying stochastic dynamics for the interest rate, as well as the market price for risk.

So far, a small number of well-known analytically tractable models have been devised by various authors. In this thesis, new tractable models are formulated in a systematic manner. First, the BPE is transformed to a standard canonical form in which only one coefficient function appears. In some interesting cases, this single coefficient function is identically zero, leaving nothing more to solve than the classical heat diffusion equation. In other cases, the canonical form allows a general solution by separation of variables. In many other cases, the general solution of the

BPE is reduced to a single inverse Laplace transform. In order to find such tractable models, consistency equations must be solved to determine suitable coefficients for the interest rate equation. These cases are classified. Many of them have desirable features such as power-law dependence of volatility on interest rates, as observed. One way of comparing fixed-income products is via the yield curve which gives a measure of how much the instrument 'earns'. For some of these models, the yield curves can be explicitly constructed.



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# Chapter 1

## Introduction

Harry Markowitz's Ph.D. thesis (1952) "Portfolio Selection" laid the groundwork for the mathematical theory of finance. By awarding Harry Markowitz, William Sharpe and Merton Miller the 1990 Nobel Prize in Economics, the Nobel Prize Committee brought to world-wide attention the fact that the last forty years have seen the emergence of a new scientific discipline, namely the "theory of finance". This theory attempts to understand how financial markets work, how to make them more efficient, and how they should be regulated.

In 1969 Robert Merton introduced stochastic calculus into the study of finance. Merton was motivated by the desire to understand how prices are set in financial markets, which is the classical economics question of "equilibrium". At the same time as Merton's work and with Merton's assistance, Fischer Black and Myron Scholes were developing their celebrated option pricing formula [4]. This work won the 1997 Nobel Prize in Economics. It provided a solution to an important practical

problem, that of finding a fair price for a European Call option, i.e., the right to buy one share of a given stock at a specified price and time. In 1981, Harrison and Pliska used the general theory of continuous-time stochastic processes to put the Black-Scholes option pricing formula on a theoretical basis, and as a result, showed how to price numerous other “derivative securities”.

Without losing its application to practical aspects of trading and regulation, the theory of finance has become increasingly mathematical, to the point that problems in finance are now driving research in mathematics. In this thesis we have looked at one such finance problem, that of providing a theoretically fair value for a simple bond. The necessary background is provided and no prior knowledge of finance is assumed.

## 1.1 Objectives

This chapter provides an introduction to the “theory of finance” and terminologies as would be deemed adequate for this thesis. As the future path of short-term interest rates is unknown, we attempt to model them as stochastic variables in order to obtain realistic results. We assume that this short-term interest rate follows a continuous random walk and has as its governing equation a stochastic differential equation. In Chapter 2 we derive the Bond Pricing Equation (BPE) and using delta hedging techniques, eliminate (from within the construction of this equation) any arbitrage possibilities. In Chapter 3 we discuss desirable properties of one-factor interest rate models and present solutions to the Bond Pricing Equation which assume various

other current interest rate models mostly of the form

$$dr = (\alpha + \beta r)dt + \sigma r^\gamma dX \quad (1.1)$$

where  $\alpha$ ,  $\beta$ ,  $\sigma$  are usually constant though in some cases are time-dependent, and  $\gamma$  is constant. The properties of these models are discussed towards the end of that chapter. By performing a comprehensive empirical analysis on one factor interest rate models of the form (1.1) Chan et al [7] found that the most successful models in capturing the dynamics of the short rate, were those that allowed the volatility of the interest rate changes to be highly sensitive to the level of the interest rate, in particular those with  $\gamma \geq 1$ . Their unconstrained estimate was  $\gamma = 1.5$ . Chapter 4, which constitutes all the original work in this thesis, lays emphasis on presenting new solutions to the Bond Pricing Equation. The forms of the coefficients are not as restricted as the other models have been. As well, in all our solutions, our short term interest rate follows a deterministic volatility of the form  $cr^2$  or  $cr^{3/2}$ .

## 1.2 Background

The most basic of financial instruments is the *equity*, *stock* or *share*. The investor (say us) could buy shares issued by a company and theoretically we would then own a certain percentage of this company, depending on how many shares we bought. To the average investor the value in holding the stock comes from the *dividends* and any growth in the stock's value both of which depend on how well the company is functioning. Dividends are periodic payments (usually quantified by so much per

unit stock) by the company to the stock holders. We now introduce some definitions which the reader may encounter within the text.

### 1.2.1 Useful terms and definitions

**Arbitrage.** The simultaneous purchase and sale of two securities that are essentially identical in order to profit from a disparity in their prices. The *Law of One Price* is that identical assets in the same market should have the same price. Most sophisticated finance theory is based on the concept of hedging and no arbitrage.

**Bears.** Investors who expect the share prices to fall.

**Bonds.** These are long term debt instruments issued by corporations or governmental entities. These are interest-only loans wherein the borrower pays interest every period and repays the principal at the end of the loan period.

**Bulls.** Investors who expect share prices to rise.

**Coupons.** These are the regular interest payments promised by the issuer.

**Coupon bearing bond.** Similar to a *zero-coupon bond* except that as well as paying the principal at maturity, it pays smaller quantities, the *coupons*, at intervals up to and including the maturity date. These coupons are usually pre-specified fractions of the principal.

**Delta Hedging.** The perfect elimination of risk by exploiting correlation between two instruments (between an option and its underlying, say). The building blocks of derivatives theory are *delta hedging* and the concept of *no arbitrage*.

**Derivative securities.** Also known as financial derivatives or derivative products.

They are contracts whose value depends on the price of a particular asset (example stocks, options, futures and forward contracts).

**Dividends.** Dividends are periodic payments (which may vary in the amount) to the owner of the stock. They provide a measure of the value of the stock.

**Drift (rate).** Measure of the average rate of growth (in our case) of the interest rate.

**Expiry (date).** Date on which an option can be exercised or date on which a bond reaches maturity. This will be denoted by  $T$ .

**Face value of a bond.** The face value of a bond is the promised amount that the issuer pays at the end of the loan.

**Hedging.** The offsetting of risk by buying other related contracts. Any reduction in risk can generally be termed hedging.

**Intrinsic value.** The payoff of an option on its expiry date that would be received if the underlying is at its current level when the option expires.

**Inverted yield curve.** A plot of yields on the vertical axis and time to maturity on the horizontal axis such that short-term rates are higher than intermediate-term rates, which are higher than long-term rates. Practitioners believe that there is often widespread misreading of the slope of yield curves as forecasting tools.

**Liquidity.** The ability to trade any asset at a price close to the current market price.

**Maturity.** Maturity refers to the time period until the face value is paid.

**Measures of yield.** There is such a variety of fixed-income products, with different

coupon structures, amortisation, fixed and/or floating rates, that it is necessary to be able to consistently compare different products. One way to do this is through measures of how much each contract earns. There are several measures of this all coming under the name *yield*. We can do this by the *current yield*, *yield to maturity* and the *price/yield relationship* to name a few.

**Par.** Stated value of stock or bond at issue.

**Perpetual bonds.** Bonds with no maturity but with a fixed stream of interest payments.

**Premium.** The amount paid for a contract initially.

**Present value.** Future receipts discounted to the present.

**Risk (components) of an interest rate.** The eleven components of risks as outlined by Parks [20] are: credit risk, inflation risk, capital risk, marketability risk, re-investment risk, liquidity risk, pure-rate risk, call risk, event risk, prepayment risk, and exchange-rate risk.

**Spot interest rate.** Also known as the *short-term interest rate*,  $r(t)$  which is the instantaneous continuously compounded interest rate at time  $t$ .

**Spot market.** Markets where the asset is delivered ‘on the spot’, or soon after, as in stock or bond markets.

**Stocks.** Equity ownership in a corporation or a financial institution.

**Time value.** Any value that the option has above its intrinsic value. The uncertainty surrounding the future value of the underlying asset means that the option value is generally different from the intrinsic value.



**Underlying(asset).** The financial instrument on which the option value depends.

**Volatility.** A measure of the standard deviation of the return.

**Wiener process.** This is a *continuous* stochastic process  $X$  whose uncorrelated increments have the property that  $dX$  is a random variable drawn from a normal distribution with mean zero and variance  $dt$ . It can be used if markets are dominated by ‘ordinary’ events while ‘extremes’ occur only infrequently, according to the probabilities in the tail areas of a normal distribution. A Wiener process is one of the basic building blocks used in modelling continuous time asset prices.

**Yield to maturity.** Yield to maturity is the rate that makes the discounted cash flows from a bond equal to its price. The spot interest rate and the yield to maturity of a zero coupon bond are same.

**Zero coupon bond.** Security issued in the primary market at a substantial discount to its principal amount at final maturity, with no coupons attached and paying no current income. They are also called *pure discount bonds* or simply *zeros*.

This section contains all the necessary financial terminology required for the remainder of this thesis.

## Chapter 2

# The Bond Pricing Equation

To assume that the interest rates are either constant or known functions of time is not sufficient to accurately model interest rates for long-term contracts, i.e. say, longer than the duration of an option. In this chapter we will be deriving the Bond Pricing Equation (BPE) which arises from the assumption that the short-term interest rate movement depends on a single source of randomness. This type of modelling is commonly referred to as *one-factor interest rate modelling*.

The spot rate,  $r$ , (or the *short rate* as it is also often called) represents the instantaneous rate of risk-less return at any time, so that \$1 invested at time  $t$  will have grown by some later time  $T$  to  $e^{\int_t^T r(u)du}$  dollars. If we let this interest rate follow a random walk (i.e. model it as a stochastic process) then we can use it as a guide for longer dated contracts as we otherwise cannot forecast the future course of an interest rate with any degree of confidence.

## 2.1 Derivation of the Bond Pricing Equation

As far as interest-rate models are concerned, the fundamental requirement of financial derivatives, that of ensuring no arbitrage possibilities, was first obtained by Vasicek in 1977 [24]. In the classic derivation, the original argument which was used, was very similar in its form to that used by Black and Scholes (1973) [4] for their work in the pricing of options.

Let us start by assuming that the interest rate is governed by a stochastic differential equation of the form

$$dr = u(r, t)dt + w(r, t)dX. \quad (2.1)$$

where  $X$  is the standard Wiener process.

The various forms that are taken by  $u$  and  $w$  determine the long term behaviour of the short rate  $r$ . Some popular forms of  $u(r, t)$  and  $w(r, t)$  will be described in Chapter 3.

Applying Itô's lemma (see e.g. [26]) to functions of  $r$  and  $t$ , gives the dynamics of the bond price as

$$dV = \left[ \frac{\partial V}{\partial r}u + \frac{\partial V}{\partial t} + \frac{w^2}{2} \frac{\partial^2 V}{\partial r^2} \right] dt + w \frac{\partial V}{\partial r} dX, \quad (2.2)$$

noting that  $V(r, t)$  represents the value of a bond when the spot rate  $r$  is governed by (2.1). Even though our primary concern at this stage is with bonds, we shall see that the resultant governing equation may be used to price other fixed income

products as well. The difference between pricing a bond and pricing an option is that as we are not modelling traded assets we do not have any underlying asset with which we can hedge. The only conceivable way by which we could overcome this difficulty is by constructing a hedged portfolio where we hedge with two bonds of different maturities, namely  $V_1(r, t; T_1)$  and  $V_2(r, t; T_2)$  where  $V_1$  and  $V_2$  are the price of bonds with maturity dates of  $T_1$  and  $T_2$  respectively.

A portfolio  $\Pi$  can then be created which contains one unit of the bond  $V_1$  and a number  $\Delta$  units of bond  $V_2$ . The value of our portfolio is thus

$$\Pi = V_1 + \Delta V_2. \quad (2.3)$$

Using (2.1) and (2.2) the process obeyed by the portfolio is then given by

$$\begin{aligned} d\Pi &= \left[ \left( \frac{\partial V_1}{\partial r} u + \frac{\partial V_1}{\partial t} + \frac{w^2}{2} \frac{\partial^2 V_1}{\partial r^2} \right)_{T=T_1} + \Delta \left( \frac{\partial V_2}{\partial r} u + \frac{\partial V_2}{\partial t} + \frac{w^2}{2} \frac{\partial^2 V_2}{\partial r^2} \right)_{T=T_2} \right] dt \\ &+ \left[ \frac{\partial V_1}{\partial r} w + \Delta \frac{\partial V_2}{\partial r} w \right] dX. \end{aligned}$$

We can easily verify by direct substitution that if  $\Delta$  is chosen to be equal to

$$\Delta = -\frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r}$$

then the portfolio (created in (2.3)) given as

$$\Pi = V_1 - \left( \frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r} \right) V_2,$$

is now purely deterministic, as with this particular choice of  $\Delta$  we have eliminated

all randomness in  $d\Pi$ . Over a small time interval  $dt$  the portfolio changes value by

$$d\Pi = dV_1 - \left( \frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r} \right) dV_2, \quad (2.4)$$

but now with this choice of  $\Delta$  our portfolio is purely deterministic and so must earn the risk-less spot interest rate, that is

$$d\Pi = r\Pi dt. \quad (2.5)$$

Equating equations (2.4) and (2.5) we get

$$dV_1 - \left( \frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r} \right) dV_2 = \left[ V_1 - \left( \frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r} \right) V_2 \right] r dt. \quad (2.6)$$

Substituting the expressions for  $dV_1$  and  $dV_2$  (obtained from (2.2)) into (2.6) we obtain

$$\left[ \left( \frac{\partial V_1}{\partial r} u + \frac{\partial V_1}{\partial t} + \frac{w^2}{2} \frac{\partial^2 V_1}{\partial r^2} \right) + \frac{\partial V_1}{\partial r} w dX \right] - \frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r} \left[ \left( \frac{\partial V_2}{\partial r} u + \frac{\partial V_2}{\partial t} + \frac{w^2}{2} \frac{\partial^2 V_2}{\partial r^2} \right) + \frac{\partial V_2}{\partial r} w dX \right] = \left[ V_1 - \frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r} V_2 \right] r dt$$

which simplifies to

$$\frac{\partial V_1}{\partial t} + \frac{w^2}{2} \frac{\partial^2 V_1}{\partial r^2} - \left( \frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r} \right) \left( \frac{\partial V_2}{\partial t} + \frac{w^2}{2} \frac{\partial^2 V_2}{\partial r^2} \right) = \left[ V_1 - \frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r} V_2 \right] r.$$

Placing all  $V_1$  terms on the left and  $V_2$  terms on the right we have

$$\frac{\frac{\partial V_1}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V_1}{\partial r^2} - r V_1}{\frac{\partial V_1}{\partial r}} = \frac{\frac{\partial V_2}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V_2}{\partial r^2} - r V_2}{\frac{\partial V_2}{\partial r}}. \quad (2.7)$$

Now (2.7) must hold true for any maturity date and the only way this will be possible is if both sides of equation (2.7) are independent of the maturity date. We set this ratio to equal some function  $\rho(r, t)$  which would be independent of the maturity date. Hence

$$\rho(r, t) = \frac{\frac{\partial V}{\partial t} + \frac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} - rV}{\frac{\partial V}{\partial r}}.$$

Writing  $\rho(r, t)$  in the form

$$\rho(r, t) = w(r, t)\lambda(r, t) - u(r, t),$$

(note that the function  $\lambda(r, t)$  is not yet specified) we obtain the following governing equation for the price of a zero coupon bond

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w) \frac{\partial V}{\partial r} - rV = 0. \quad (2.8)$$

We hence refer to equation (2.8) as the Bond Pricing Equation. Equation (2.8) could also be seen as a Convection-Diffusion-Reaction equation. The first two terms constitute the basic diffusion equation. The first-order  $r$ -derivative term

$$(u - \lambda w) \frac{\partial V}{\partial r}$$

can be thought of as a convection term and the final term

$$-rV$$

as a reaction term. A similar Convection-Diffusion-Reaction equation is used to model the flow of solutes in soils where the diffusive component constitutes the diffusion of the solute in the soil water, the convective term corresponds to the advection of the solute by the water within the soil pores and the reactive term would highlight any biological or chemical reaction of the pollutant within the soil matrix.

The bond pricing equation is a backward equation as the diffusion term is unstable in the forward direction. Hence we have to impose a final condition on our solution. The final condition corresponds to the payoff at maturity and so for a zero-coupon bond we set

$$V(r, T; T) = 1.$$

For convenience, the specific value at maturity is taken to be one unit. This specification has no bearing on the form of solution of the linear boundary value problem. To find a unique solution to (2.8) we would need to impose two boundary conditions. These boundary conditions would depend on the form of  $u(r, t)$  and  $w(r, t)$ .

Now that we have derived the Bond Pricing Equation (BPE), we note that no specific condition was set on  $V(r, t)$  to represent the bond price; it was merely notation. We can hence safely assume that this same partial differential equation (with a specified final condition) would have to be obeyed by any traded asset if we were to ensure no arbitrage possibilities.

## 2.2 The market price of risk

Equation (2.2) may be rewritten (using (2.8)) as

$$dV - rVdt = w \frac{\partial V}{\partial r} (dX + \lambda dt). \quad (2.9)$$

Due to the presence of the  $dX$  term we can see that equation (2.9) is not purely deterministic. The  $\lambda dt$  term in (2.9) can be viewed as the extra profit a portfolio stands to make per unit of extra risk the portfolio takes, therefore the function  $\lambda$  is called the *market price of risk*. If the investors were risk-averse then  $\lambda$  would be positive, if they were risk-neutral then  $\lambda$  would be zero and if they were risk-seeking then  $\lambda$  would be negative. We may also write  $\lambda(r, t)$  from (2.8) as

$$\lambda(r, t) = \frac{\frac{\partial V}{\partial r}u + \frac{\partial V}{\partial t} + \frac{w^2}{2} \frac{\partial^2 V}{\partial r^2} - rV}{w \frac{\partial V}{\partial r}}. \quad (2.10)$$

From equation (2.10) we see that the *difference* between the real-world drift of the price of a bond (see (2.2))

$$\frac{\partial V}{\partial r}u + \frac{\partial V}{\partial t} + \frac{w^2}{2} \frac{\partial^2 V}{\partial r^2}$$

and the risk-neutral drift  $rV$ , divided by the absolute volatility of the bond price  $w \frac{\partial V}{\partial r}$  (see (2.2)) is what investors interpret, quite justifiably, as the market price of risk, which enables them to make the extra ‘profit’ on their portfolio for the extra ‘risk’ they take. If investors were risk-neutral,  $\lambda$  would be zero and hence the portfolio would not profit from the risk taken.



Note that in the Bond Pricing Equation (2.8),  $u - \lambda w$  is the coefficient of  $\frac{\partial V}{\partial r}$  and that  $w$  appears in the coefficient of  $\frac{\partial^2 V}{\partial r^2}$ . One interpretation (as Wilmott [26] quite appropriately puts it) of the solution to the Bond Pricing Equation is that it can be seen as the expected present value of all cash flows. This expectation is with respect to the risk-neutral variable and not with respect to the real random variable. This is so because the drift term in the equation is the drift of the risk-neutral spot rate, which has a drift of  $u - \lambda w$ . Note that the drift term in the equation is not just the drift of the real spot rate which is  $u$ .

Hopefully this chapter has provided us with adequate background knowledge about the equations that we will be dealing with. In the next chapter we introduce some known tractable models of the Bond Pricing Equation.

# Chapter 3

## Solutions to the Bond Pricing Equation

In the previous chapter we derived the Bond Pricing Equation as an arbitrary model in the sense that we did not specify any functional forms for the risk-neutral drift,  $u - \lambda w$ , or for the volatility,  $w$ . The questions then remain as to how we can choose these functions to give us a good model, and what properties we would like our model to exhibit. In this chapter we firstly look at desired features for our one-factor models and discuss them in reference to various existing models that we outline in this chapter. We also look at specific forms for the risk-neutral drift and the volatility that lead to tractable models.

### 3.1 Desired properties of equation (2.1)

*Mean reversion:* In historical terms, very high values of interest rates are usually followed by a decrease in the rates more frequently than by an increase. Similarly, very low rates tend to be followed by an increase more frequently than a decrease. This behaviour is suggestive of a mean reverting process. As an example of a mean reverting process consider

$$dx = \lambda(\mu - x)dt + \sigma dX.$$

If  $x$  falls below the mean value  $\mu$ , then  $\mu - x$  will become positive. So then  $dx$  will be more likely to stay positive and  $x$  will eventually move towards  $\mu$ , i.e. it will tend to revert back to its mean. Note that this is an example of an Ornstein-Uhlenbeck process, where  $\mu$  is the mean reverting level, with  $\lambda$  referred to as the speed of mean reversion.

*Positive interest rates:* We would prefer interest rates not to become negative or even to assume very large values as this hardly ever happens in reality. I say ‘hardly ever’ as in the 1960s Switzerland witnessed non-positive interest rates, which indeed is a rare event. As well, with suitable choice of parameters, we can ensure that the interest rate stays positive.

However, as we later discuss in Section 3.5, models that do not have all the desired properties should not necessarily be discarded.

## 3.2 Tractable models

Tractable models are models for which closed form solutions of the Bond Pricing Equation (for zero-coupon bonds) can be found analytically. Let us assume that the random walk for  $r$  (the risk-neutral rate) is again

$$dr = (u - \lambda w)dt + wdX, \quad (3.1)$$

where  $X$  is a standard Wiener process,

$$u(r, t) - \lambda(r, t)w(r, t) = \eta(t) - \gamma(t)r, \quad (3.2)$$

and

$$w(r, t) = \sqrt{\alpha(t)r + \beta(t)}. \quad (3.3)$$

Here we are describing a model for the risk-neutral spot rate. With the drift term (3.2), our random walk model (3.1) for  $r$  has the property of mean reversion. For large  $r$  the (risk-neutral) interest rate will tend to decrease towards the mean, which may be a function of time and when the rate is small, it will on average move upwards. In the above model the spot rate is bounded below by a positive number if  $\alpha(t) > 0$  and  $\beta \leq 0$  giving the lower bound as  $-\beta/\alpha$  [26]. If  $\alpha(t) = 0$  then we must have  $\beta(t) \geq 0$ .

By choosing  $u(r, t)$  and  $w(r, t)$  in the stochastic differential equation for  $r$  to take the special functional forms (3.2) and (3.3) the BPE (2.8) (for a zero-coupon bond) becomes

$$\frac{\partial Z}{\partial t} + \frac{1}{2}(\alpha(t)r + \beta(t))\frac{\partial^2 Z}{\partial r^2} + (\eta(t) - \gamma(t)r)\frac{\partial Z}{\partial r} - rZ = 0. \quad (3.4)$$

The solution is then of the form

$$Z(r, t; T) = e^{A(t; T) - rB(t; T)}, \quad (3.5)$$

where  $Z(r, t; T)$  denotes the value of a zero-coupon bond and where  $A$  and  $B$  need to satisfy

$$\frac{\partial A}{\partial t} = \eta(t)B - \frac{1}{2}\beta(t)B^2 \quad (3.6)$$

and

$$\frac{\partial B}{\partial t} = \frac{1}{2}\alpha(t)B^2 + \gamma(t)B - 1. \quad (3.7)$$

respectively.

It has been shown (independently by Duffie (1992) [9], Klugman (1992) [15], Klugman & Wilmott (1994) [16] and others) that if the solution to (3.4) for the zero-coupon bond takes the form (3.5) then the most general forms for the coefficients take the form given by (3.2) and (3.3).

In order to satisfy the final condition that  $Z(r, T; T) = 1$  we must have

$$A(T; T) = 0 \text{ and } B(T; T) = 0.$$

### 3.3 Solution for constant parameters

The solution for arbitrary  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\eta$  is found by integrating the two ordinary differential equations (3.6) and (3.7). However it is not always possible to integrate these equations explicitly. The simplest case is when  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\eta$  are all constant, in which case the stochastic differential equation that  $r$  satisfies takes the form

$$dr = (\eta - \gamma r)dt + \sqrt{(\alpha r + \beta)}dX,$$

and the Bond Pricing Equation takes the form

$$\frac{\partial Z}{\partial t} + \frac{1}{2}(\alpha r + \beta)\frac{\partial^2 Z}{\partial r^2} + (\eta - \gamma r)\frac{\partial Z}{\partial r} - rZ = 0,$$

which has a solution [26] of the form

$$Z(r, t; T) = e^{A(t; T) - rB(t; T)},$$

where

$$\frac{\alpha}{2}A = a\psi_2 \log(a - B) + (\psi_2 + \frac{1}{2}\beta)b \log((B + b)/b) - \frac{1}{2}B\beta - a\psi_2 \log(a) \quad (3.8)$$

and

$$B = \frac{2(e^{\psi_1(T-t)} - 1)}{(\gamma + \psi_1)(e^{\psi_1(T-t)} - 1) + 2\psi_1}, \quad (3.9)$$

and where

$$b, a = \frac{\pm\gamma + \sqrt{\gamma^2 + 2\alpha}}{\alpha},$$

$$\psi_1 = \sqrt{\gamma^2 + 2\alpha} \quad \text{and} \quad \psi_2 = \frac{\eta - a\beta/2}{a + b}.$$

Some graphs have been plotted for this solution for expiry times of 2 and 5 years for various interest rates. Figure (3.1) and Figure (3.2) show the change in the value of the bond for the same interest rates but with expiry times of  $T = 2$  years and  $T = 5$  years respectively. Bonds with lower interest rates are 'worth' more than the ones with higher rates at the same specific time during the duration of the bond, except at maturity where the payoff is the same.

One way to compare fixed income products is to get a relative measure of the amount they earn, which is often done in interest rate modelling via the *yield curve*. Let us suppose that we have a zero-coupon bond  $Z(r, t; T)$  which pays \$ 1 at maturity, i.e. at  $t = T$ . Now if we apply a constant rate of return of  $y$  between  $t$  and  $T$ , then \$ 1 received at time  $T$  has a present value of  $Z(t; T)$  at time  $t$ , where

$$Z(t; T) = e^{-y(T-t)},$$

giving

$$y = -\frac{\log(Z)}{T-t}.$$

The yield curve is the plot of  $y$  against time to maturity  $T-t$ . In fact the dependence of the yield curve on the time to maturity is often referred to as the *term structure of interest rates*. An increasing yield curve in economic terms implies that future interest rates are higher than the short-term rate, while a decreasing yield curve is typical of periods when the short-term rate is high but is expected to fall and finally we often get the humped yield curve which again signifies a fall in the short rate. We

shall be plotting these yield curves for some of the specific standard models which we introduce in the next section.

So far we have introduced tractable models, and noted that it is desirable to have the functional forms of our choice of the parameters to lead to the properties of mean reversion and positive interest rates. The solution of the BPE with constant parameters has also been given.



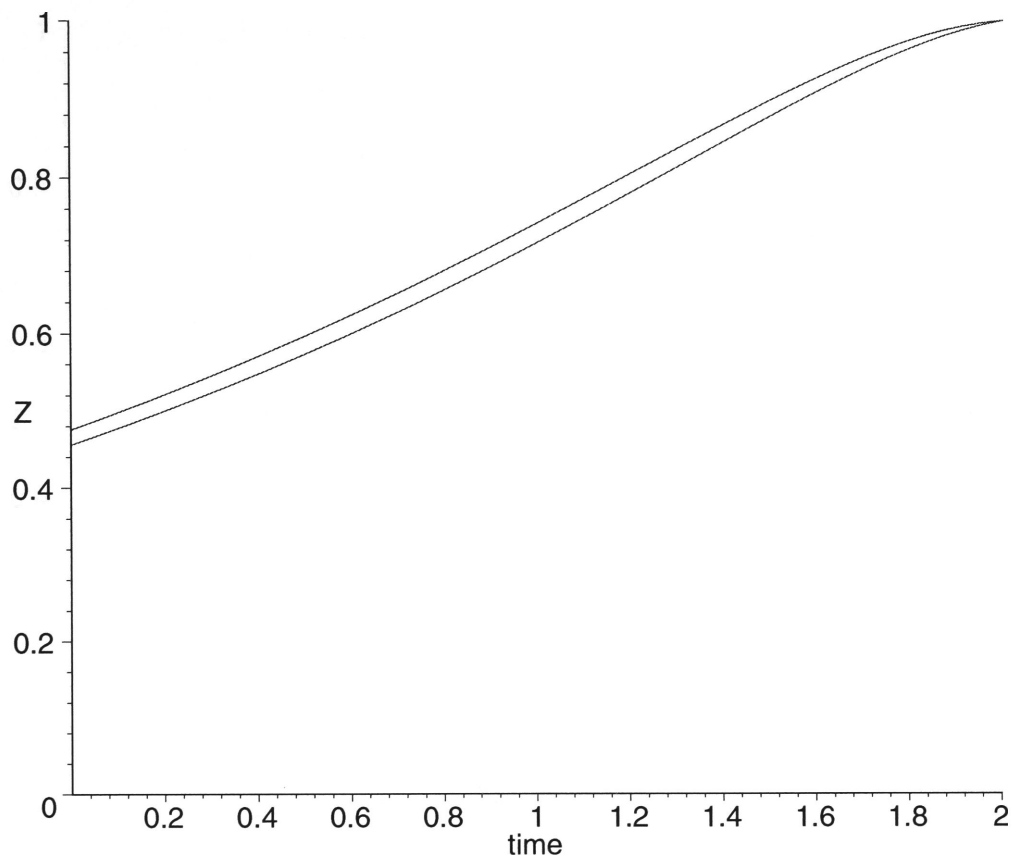


Figure 3.1: Sample solution for constant parameters with  $r = 4\%$  (top curve) and  $r = 10\%$  (bottom curve) respectively with time to maturity  $T = 2$  years. Parameter values:  $\alpha = 1, \beta = 1, \eta = 1, \gamma = 1$

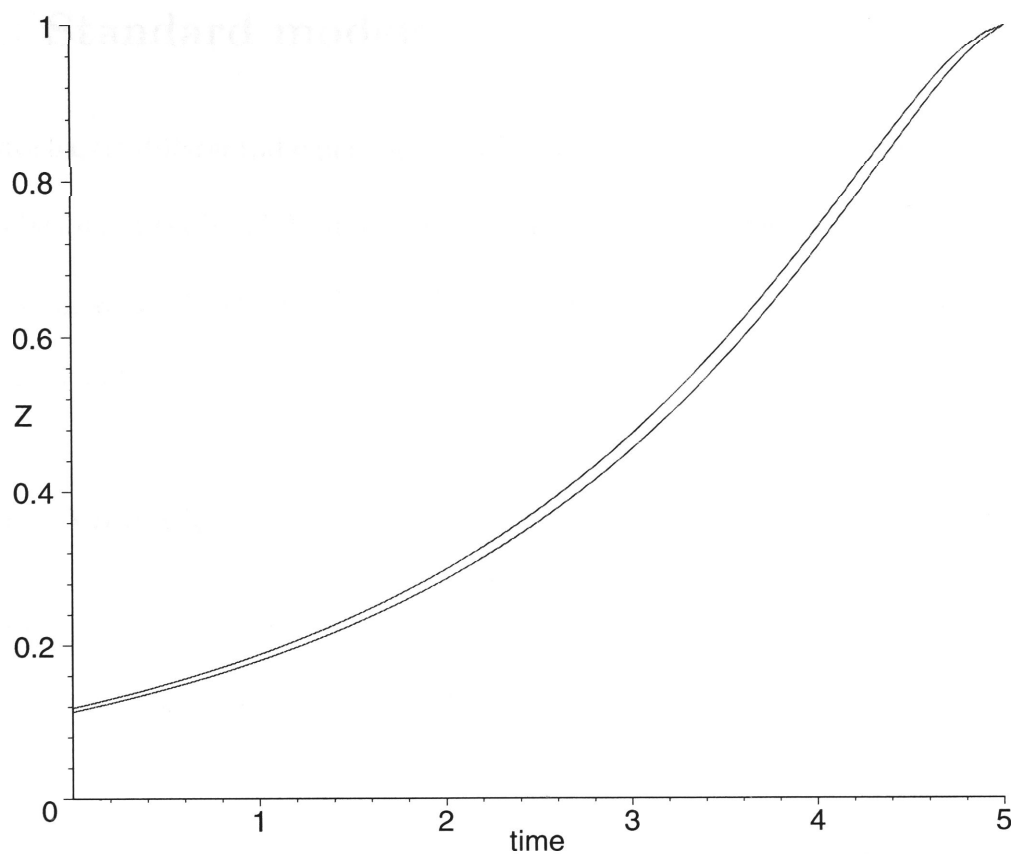


Figure 3.2: Sample solution for constant parameters with  $r = 4\%$  (top curve) and  $r = 10\%$  (bottom curve) respectively with time to maturity  $T = 5$  years. Parameter values:  $\alpha = 1, \beta = 1, \eta = 1, \gamma = 1$

## 3.4 Standard models

The stochastic differential equation (3.1) for the interest rate process, with the drift and volatility given by (3.2) and (3.3), incorporates the models of Vasicek [24], Cox, Ingersoll & Ross [8], Ho and Lee [12], and Hull & White [13]. We now outline the Vasicek model.

### 3.4.1 Vasicek

The Vasicek model [24] for the interest rate takes the form of (3.2) and (3.3) with  $\alpha = 0$ ,  $\beta > 0$  and with all other parameters independent of time, giving (3.1) as

$$dr = (\eta - \gamma r)dt + \beta^{1/2}dX,$$

and hence the Bond Pricing Equation as

$$\frac{\partial V}{\partial t} + \frac{\beta}{2} \frac{\partial^2 V}{\partial r^2} + (\eta - \gamma r) \frac{\partial V}{\partial r} - rV = 0.$$

Analytic solutions for this model can be found and the values of a zero-coupon bond (satisfying the condition  $Z(r, T; T) = 1$ ) are given by

$$Z(r, t; T) = e^{A(t; T) - rB(t; T)},$$

where

$$B(t; T) = \frac{1}{\gamma} (1 - e^{\gamma(T-t)}),$$

and

$$A(t; T) = \frac{1}{\gamma^2} (B(t; T) - T + t)(\eta\gamma - \frac{1}{2}\beta) - \frac{\beta B(t; T)^2}{4\gamma}.$$

The Vasicek model is in fact just a special case of the constant parameter model as the functions appearing in (3.2) and (3.3) are independent of time.

### 3.4.2 Cox, Ingersoll and Ross

The CIR model [8] takes (3.2) and (3.3) as the interest rate model but with  $\beta = 0$ , and again no time dependence in the parameters giving (3.1) as

$$dr = (\eta - \gamma r)dt + \sqrt{\alpha r}dX.$$

The Bond Pricing Equation then takes the form

$$\frac{\partial V}{\partial t} + \frac{\alpha r}{2} \frac{\partial^2 V}{\partial r^2} + (\eta - \gamma r) \frac{\partial V}{\partial r} - rV = 0.$$

A solution to this equation satisfying the final condition  $Z(r, T; T) = 1$  is given as

$$Z(r, t; T) = e^{A(t; T) - rB(t; T)},$$

where

$$A = \frac{2\psi_2}{\alpha} \{a \log(a - B) + b \log((B + b)/b) - a \log(a)\},$$

and

$$B = \frac{2(e^{\psi_1(T-t)} - 1)}{(\gamma + \psi_1)(e^{\psi_1(T-t)} - 1) + 2\psi_1},$$

and where

$$b, a = \frac{\pm\gamma + \sqrt{\gamma^2 + 2\alpha}}{\alpha},$$

$$\psi_1 = \sqrt{\gamma^2 + 2\alpha} \text{ and } \psi_2 = \frac{\eta}{a + b}.$$

Note that the resulting expressions are not really simpler than the non-zero  $\beta$  solution as outlined in (3.8) and (3.9).

In this model the spot rate process is mean reverting and if  $\eta > \alpha/2$  the interest rate stays positive [8]. Figure (3.3) shows a sample change in the value of the bond (using the CIR model) and (3.4) shows the corresponding yield curve.

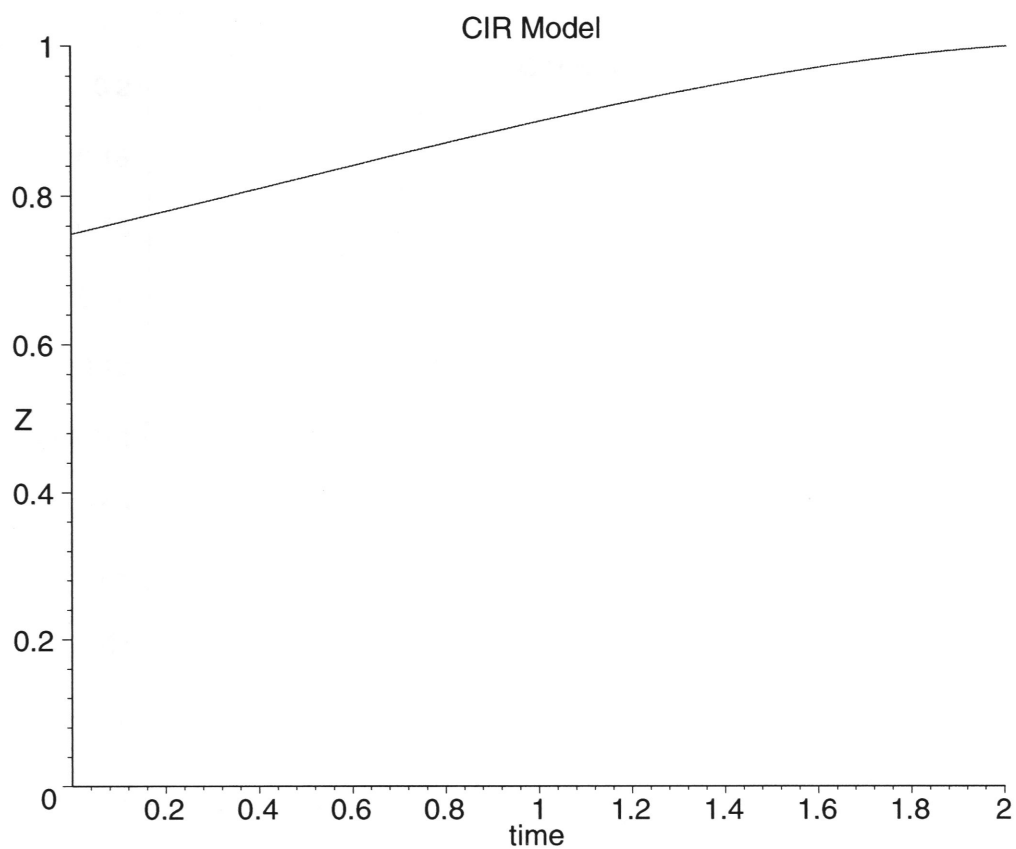


Figure 3.3: Sample bond price solution of the CIR model. Parameter values:  $T = 2, r = 0.04, \alpha = 0.0225, \eta = 0.2, \gamma = 0.8$

## 3.4.3. Ho and Lee

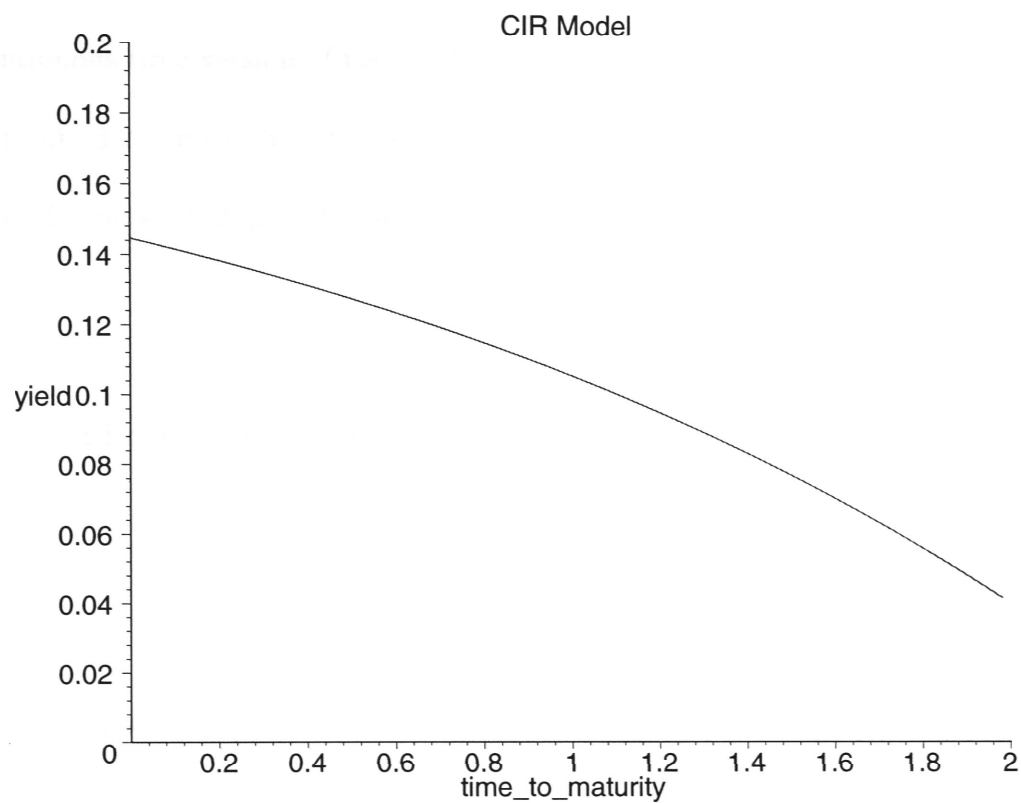


Figure 3.4: Sample yield curve for the CIR model. Parameter values:  $T = 2$ ,  $r = 0.04$ ,  $\alpha = 0.0225$ ,  $\eta = 0.2$ ,  $\gamma = 0.8$

### 3.4.3 Ho and Lee

The continuous time version of the Ho & Lee model [12] for the interest rate, takes the form of (3.2) and (3.3) with  $\alpha = \gamma = 0$ ,  $\beta > 0$  and constant and with  $\eta$  a function of time so that (3.1) becomes

$$dr = \eta(t)dt + cdX.$$

The Bond Pricing Equation then is

$$\frac{\partial V}{\partial t} + \frac{c^2}{2} \frac{\partial^2 V}{\partial r^2} + \eta(t) \frac{\partial V}{\partial r} - rV = 0.$$

The standard deviation of the spot rate process,  $c$ , is constant and the drift rate  $\eta$  is time dependent. The value of the zero-coupon bonds is given by

$$Z(r, t; T) = e^{A(t; T) - rB(t; T)},$$

where

$$B = T - t,$$

and

$$A = - \int_t^T \eta(s)(T - s)ds + \frac{1}{6}c^2(T - t)^3.$$

If we know  $\eta(t)$  then the above solution gives us the theoretical value of zero-coupon bonds. A careful choice for the function  $\eta(t)$  will result in theoretical zero-coupon bond prices, as given by the model, to match the market prices. This technique of choosing appropriate functions to match theoretical and market values



is called *yield curve fitting*. For this model the careful choice is (see Wilmott [26] for details)

$$\eta(t) = -\frac{\partial^2}{\partial t^2} \log Z_M(t^*; t) + c^2(t - t^*)$$

where today is time  $t = t^*$  and where  $Z_M(t^*; T)$  denotes the market price today of zero-coupon bonds with maturity  $T$ .

With this choice for the time-dependent parameter  $\eta(t)$  the theoretical and the actual market prices of zero-coupon bonds are the same and it follows that

$$A(t; T) = \log \left( \frac{Z_M(t^*; T)}{Z_M(t^*; t)} \right) - (T - t) \frac{\partial}{\partial t} \log(Z_M(t^*; t)) - \frac{1}{2} c^2 (t - t^*) (T - t)^2.$$

The Ho & Lee model can fit the yield curve to the market prices which is an additional advantage of the model, but as the diffusion term in the equation does not depend on  $r$  the model may unfortunately allow the interest rates to go negative [18].

### 3.4.4 Hull and White

Hull & White [13] have extended both the Vasicek and the CIR models to incorporate time-dependent parameters. This time dependence allows the yield curve to be fitted. Recall that the Vasicek model had

$$dr = (\eta - \gamma r)dt + cdX,$$

as the stochastic differential equation for the spot rate process. Hull & White extended this to include a time-dependent parameter giving

$$dr = (\eta(t) - \gamma r)dt + c dX.$$

Again the value of zero-coupon bonds takes the form

$$Z(r, t; T) = e^{A(t; T) - rB(t; T)}$$

where

$$A(t; T) = - \int_t^T \eta^*(s) B(s; T) ds + \frac{c^2}{2\gamma^2} \left( T - t + \frac{2}{\gamma} e^{-\gamma(T-t)} - \frac{1}{2\gamma} e^{-2\gamma(T-t)} - \frac{3}{2\gamma} \right),$$

and

$$B(t; T) = \frac{1}{\gamma} (1 - e^{-\gamma(T-t)}).$$

To fit the yield curve at time  $t^*$  we must make  $\eta^*(t)$  satisfy

$$\begin{aligned} A(t^*; T) &= - \int_{t^*}^T \eta^*(s) B(s; T) ds + \frac{c^2}{2\gamma^2} \left( T - t^* + \frac{2}{\gamma} e^{-\gamma(T-t^*)} - \frac{1}{2\gamma} e^{-2\gamma(T-t^*)} - \frac{3}{2\gamma} \right) \\ &= \log(Z_M(t^*; T)) + r^* B(t^*; T). \end{aligned} \quad (3.10)$$

We can get the form of  $\eta^*(t)$  by differentiating (3.10) twice with respect to  $T$  giving

us

$$\eta^*(t) = -\frac{\partial^2}{\partial t^2} \log(Z_M(t^*; t)) - \gamma \frac{\partial}{\partial t} \log(Z_M(t^*; t)) + \frac{c^2}{2\gamma} (1 - e^{-2\gamma(t-t^*)}).$$

$A(t; T)$  is now given by

$$A(t; T) = \log \left( \frac{Z_M(t^*; T)}{Z_M(t^*; t)} \right) - B(t; T) \frac{\partial}{\partial t} \log(Z_M(t^*; t)) - \frac{c^2}{4\gamma^3} (e^{-\gamma(T-t^*)} - e^{-\gamma(t-t^*)})(e^{2\gamma(t-t^*)} - 1).$$

Even with the extension of the Vasicek model by Hull & White, the volatility term in the SDE for the spot rate process does not depend on  $r$  and can hence lead to negative values for the interest rates. Hull & White also extended the CIR model by including time dependent parameters as they did with the Vasicek model.

Some other models for the short-term interest rate include the continuous time version of Black, Derman & Toy (BDT) [2]. In their model they have

$$d(\ln r) = \left( \theta(t) - \frac{\sigma'(t)}{\sigma(t)} \ln r \right) dt + \sigma(t) dX.$$

A more general model is the one proposed by Black & Karasinski [3] where the governing equation for the spot rate process is given by

$$d(\ln r) = (\theta(t) - a(t) \ln r) dt + \sigma(t) dX.$$

These models are quite popular because fitting can be done by a numerical scheme, however there are no explicit analytic solutions to the BPE for these models.

### 3.5 Discussion

In this chapter, solutions to and properties of, some well-known standard models of the spot rate were given and discussed. In these models the simple forms of the

coefficients gave rise to analytic solutions to the Bond Pricing Equation, so that we could easily derive the whole yield curve.

In a nutshell the desired properties for the models of the spot rate were that they should not allow the interest rates to go negative and should display a mean reverting process. The ability to fit yield curves using present data would be an additional advantage. We noticed that only with the CIR model (with  $\eta > \gamma/2$ ) did we have any form of guarantee that the interest rate would stay positive. On the other hand the Ho & Lee and the extended Vasicek model by Hull & White could fit yield curves. Does that mean that a model not exhibiting *all* the ‘desired’ qualities should necessarily find its way into the recycling bin?

It is known that no one-factor or multi-factor interest rate model can incorporate all the features of an ‘ideal’ model. According to Rebonato [22] what is really important is to identify the essential features and also the features that we can do without, for a given application.

There are also mixed feelings about the ability of a model to fit the yield curve. Practitioners argue in favour of calibration, but face criticism from a modelling perspective. Practitioners feel that one cannot hedge with any product if the market price is very different from the theoretical price and they actually try to fit as many properties as they can.

Now assume that we have found parameters, including our time-dependent parameter  $\eta(t)$ , so that the market price of simple bonds is accurately given by our model (such as the Ho & Lee model) at time  $t = t^*$ . This model though, would only

be strictly valid if the parameters were to remain unchanged when we come to re-fit them, in say  $t$  days time. This however is unlikely to the case.

In the next chapter we develop new analytic solutions for the Bond Pricing Equation. We do not restrict the drift rate and the volatility terms in (2.1) to be given by (3.2) and (3.3) but keep them as arbitrary functions of  $t$  and the spot rate  $r$ , hence our solutions do not have the form

$$e^{A(t;T)-rB(t;T)}$$

which the above described models have.

# Chapter 4

## New analytic solutions to the Bond Pricing Equation

In Chapter 2 we used a one-factor stochastic model for spot interest rates to derive a parabolic partial differential equation for the pricing of bonds and other interest rate derivative products. Then in Chapter 3 we briefly described the named models and the solutions to the BPE associated with these models. Currently there are several popular models of the spot rate process, as we showed in Chapter 3, that lead to tractable models of the Bond Pricing Equation. These models are useful as they can be used to derive the whole yield curve. This chapter is organised as follows: In Section 4.1 we first reduce the Bond Pricing Equation (2.8) to its canonical form (using symmetry methods [5]) and then in Section 4.2 (after some preliminary simplification) construct new analytical solutions using *(i)* Separation of variables (Section 4.2.1) and *(ii)* Laplace transforms (Section 4.2.2). And finally

we facilitate analytical solutions to the BPE by reducing it to the classical heat equation in Section 4.2.3.

## 4.1 Reduction to canonical form

It is well-known (e.g. Bluman and Kumei [5]) that for any parabolic equation

$$\frac{\partial^2 v}{\partial x^2} + \alpha(x, y) \frac{\partial v}{\partial x} + \beta(x, y) \frac{\partial v}{\partial y} + \gamma(x, y)v = 0, \quad (4.1)$$

there exists a point transformation of the form

$$\begin{aligned} x_1 &= x_1(x, y), \\ x_2 &= x_2(x, y), \text{ and} \\ z &= H(x, y)v, \end{aligned}$$

such that (4.1) becomes

$$\frac{\partial^2 z}{\partial x_1^2} + \frac{\partial z}{\partial x_2} + Q(x_1, x_2)z = 0, \quad (4.2)$$

for some function  $Q(x_1, x_2)$ . This canonical form is much more convenient for analysis, since it includes only one model-dependent adjustable coefficient function. We shall find special cases of the coefficient function  $Q(x_1, x_2)$  that allow either a full general solution of (4.2) or reduction to an inverse Laplace transform. These special forms of the coefficient function  $Q$  need to correspond to reasonable forms for the

Bond Pricing Equation. Hence the required transformation to (4.2) must now be considered in close detail.

Recall the BPE ,

$$\frac{\partial V}{\partial t} + \frac{w^2}{2} \frac{\partial^2 V}{\partial r^2} + (u - \lambda w) \frac{\partial V}{\partial r} - rV = 0, \quad (4.3)$$

where  $w = w(r, t)$ ,  $u = u(r, t)$  and  $\lambda = \lambda(r, t)$ . This equation can be rewritten as

$$\frac{\partial^2 V}{\partial r^2} + \alpha(r, t) \frac{\partial V}{\partial r} + \beta(r, t) \frac{\partial V}{\partial t} + \gamma(r, t) V = 0, \quad (4.4)$$

where

$$\begin{aligned} \alpha(r, t) &= 2 \left( \frac{u - \lambda w}{w^2} \right), \\ \beta(r, t) &= \frac{2}{w^2}, \text{ and} \\ \gamma(r, t) &= \frac{-2r}{w^2}, \end{aligned}$$

noting that  $w = w(r, t) \neq 0$ ,  $u = u(r, t)$  and  $\lambda = \lambda(r, t)$ .

Hence by substituting the point transformations

$$r_1 = r_1(r, t), \quad r_2 = r_2(r, t) \text{ and } z = H(r, t)V$$

into

$$\frac{\partial^2 z}{\partial r_1^2} + \frac{\partial z}{\partial r_2} + Q(r_1, r_2)z = 0, \quad (4.5)$$



and then equating the coefficients of the derivative terms of the resultant equation with those of equation (4.4) we can obtain the transformations that reduce (4.4) to (4.5).

By the product rule we have

$$\frac{\partial z}{\partial r_1} = \frac{\partial}{\partial r} [H(r, t)V] \frac{\partial r}{\partial r_1} + \frac{\partial}{\partial t} [H(r, t)V] \frac{\partial t}{\partial r_1},$$

giving us

$$\frac{\partial z}{\partial r_1} = V \frac{\partial H}{\partial r} \frac{\partial r}{\partial r_1} + H \frac{\partial V}{\partial r} \frac{\partial r}{\partial r_1} + V \frac{\partial H}{\partial t} \frac{\partial t}{\partial r_1} + H \frac{\partial V}{\partial t} \frac{\partial t}{\partial r_1}.$$

Similarly we get

$$\frac{\partial z}{\partial r_2} = V \frac{\partial H}{\partial r} \frac{\partial r}{\partial r_2} + H \frac{\partial V}{\partial r} \frac{\partial r}{\partial r_2} + V \frac{\partial H}{\partial t} \frac{\partial t}{\partial r_2} + H \frac{\partial V}{\partial t} \frac{\partial t}{\partial r_2}. \quad (4.6)$$

Also by the chain rule

$$\frac{\partial^2 z}{\partial r_1^2} = \frac{\partial}{\partial r_1} \left[ \frac{\partial z}{\partial r_1} \right] = \frac{\partial}{\partial r} \left[ \frac{\partial z}{\partial r_1} \right] \frac{\partial r}{\partial r_1} + \frac{\partial}{\partial t} \left[ \frac{\partial z}{\partial r_1} \right] \frac{\partial t}{\partial r_1},$$

giving us

$$\begin{aligned} \frac{\partial^2 z}{\partial r_1^2} &= \frac{\partial}{\partial r} \left[ V \frac{\partial H}{\partial r} \frac{\partial r}{\partial r_1} + H \frac{\partial V}{\partial r} \frac{\partial r}{\partial r_1} + V \frac{\partial H}{\partial t} \frac{\partial t}{\partial r_1} + H \frac{\partial V}{\partial t} \frac{\partial t}{\partial r_1} \right] \frac{\partial r}{\partial r_1} \\ &+ \frac{\partial}{\partial t} \left[ V \frac{\partial H}{\partial r} \frac{\partial r}{\partial r_1} + H \frac{\partial V}{\partial r} \frac{\partial r}{\partial r_1} + V \frac{\partial H}{\partial t} \frac{\partial t}{\partial r_1} + H \frac{\partial V}{\partial t} \frac{\partial t}{\partial r_1} \right] \frac{\partial t}{\partial r_1}. \end{aligned}$$

Expanding by the chain rule we get

$$\begin{aligned}
\frac{\partial^2 z}{\partial r_1^2} = & \left[ \frac{\partial}{\partial r} \left( V \frac{\partial H}{\partial r} \right) \frac{\partial r}{\partial r_1} + V \frac{\partial H}{\partial r} \frac{\partial}{\partial r} \left( \frac{\partial r}{\partial r_1} \right) \right] \frac{\partial r}{\partial r_1} + \\
& + \left[ \frac{\partial}{\partial r} \left( H \frac{\partial V}{\partial r} \right) \frac{\partial r}{\partial r_1} + H \frac{\partial V}{\partial r} \frac{\partial}{\partial r} \left( \frac{\partial r}{\partial r_1} \right) \right] \frac{\partial r}{\partial r_1} + \\
& + \left[ \frac{\partial}{\partial r} \left( V \frac{\partial H}{\partial t} \right) \frac{\partial t}{\partial r_1} + V \frac{\partial H}{\partial t} \frac{\partial}{\partial r} \left( \frac{\partial t}{\partial r_1} \right) \right] \frac{\partial r}{\partial r_1} + \\
& + \left[ \frac{\partial}{\partial r} \left( H \frac{\partial V}{\partial t} \right) \frac{\partial t}{\partial r_1} + H \frac{\partial V}{\partial t} \frac{\partial}{\partial r} \left( \frac{\partial t}{\partial r_1} \right) \right] \frac{\partial r}{\partial r_1} + \\
& + \left[ \frac{\partial}{\partial t} \left( V \frac{\partial H}{\partial r} \right) \frac{\partial r}{\partial r_1} + V \frac{\partial H}{\partial r} \frac{\partial}{\partial t} \left( \frac{\partial r}{\partial r_1} \right) \right] \frac{\partial t}{\partial r_1} + \\
& + \left[ \frac{\partial}{\partial t} \left( H \frac{\partial V}{\partial r} \right) \frac{\partial r}{\partial r_1} + H \frac{\partial V}{\partial r} \frac{\partial}{\partial t} \left( \frac{\partial r}{\partial r_1} \right) \right] \frac{\partial t}{\partial r_1} + \\
& + \left[ \frac{\partial}{\partial t} \left( V \frac{\partial H}{\partial t} \right) \frac{\partial t}{\partial r_1} + V \frac{\partial H}{\partial t} \frac{\partial}{\partial t} \left( \frac{\partial t}{\partial r_1} \right) \right] \frac{\partial t}{\partial r_1} + \\
& + \left[ \frac{\partial}{\partial t} \left( H \frac{\partial V}{\partial t} \right) \frac{\partial t}{\partial r_1} + H \frac{\partial V}{\partial t} \frac{\partial}{\partial t} \left( \frac{\partial t}{\partial r_1} \right) \right] \frac{\partial t}{\partial r_1}. \tag{4.7}
\end{aligned}$$

In (4.7) notice that

$$\begin{aligned}
V \frac{\partial H}{\partial r} \frac{\partial}{\partial r} \left( \frac{\partial r}{\partial r_1} \right) \frac{\partial r}{\partial r_1} + V \frac{\partial H}{\partial r} \frac{\partial}{\partial t} \left( \frac{\partial r}{\partial r_1} \right) \frac{\partial t}{\partial r_1} &= V \frac{\partial H}{\partial r} \left[ \frac{\partial}{\partial r} \left( \frac{\partial r}{\partial r_1} \right) \frac{\partial r}{\partial r_1} + \frac{\partial}{\partial t} \left( \frac{\partial r}{\partial r_1} \right) \frac{\partial t}{\partial r_1} \right] \\
&= V \frac{\partial H}{\partial r} \left[ \frac{\partial}{\partial r_1} \left( \frac{\partial r}{\partial r_1} \right) \right] = V \frac{\partial H}{\partial r} \frac{\partial^2 r}{\partial r_1^2}
\end{aligned}$$

Therefore

$$V \frac{\partial H}{\partial r} \frac{\partial}{\partial r} \left( \frac{\partial r}{\partial r_1} \right) \frac{\partial r}{\partial r_1} + V \frac{\partial H}{\partial r} \frac{\partial}{\partial t} \left( \frac{\partial r}{\partial r_1} \right) \frac{\partial t}{\partial r_1} = V \frac{\partial H}{\partial r} \frac{\partial^2 r}{\partial r_1^2}, \tag{4.8}$$

and similarly

$$H \frac{\partial V}{\partial r} \frac{\partial}{\partial r} \left( \frac{\partial r}{\partial r_1} \right) \frac{\partial r}{\partial r_1} + H \frac{\partial V}{\partial r} \frac{\partial}{\partial t} \left( \frac{\partial r}{\partial r_1} \right) \frac{\partial t}{\partial r_1} = H \frac{\partial V}{\partial r} \frac{\partial^2 r}{\partial r_1^2}, \tag{4.9}$$

$$V \frac{\partial H}{\partial t} \frac{\partial}{\partial r} \left( \frac{\partial t}{\partial r_1} \right) \frac{\partial r}{\partial r_1} + V \frac{\partial H}{\partial t} \frac{\partial}{\partial t} \left( \frac{\partial t}{\partial r_1} \right) \frac{\partial t}{\partial r_1} = V \frac{\partial H}{\partial t} \frac{\partial^2 t}{\partial r_1^2}, \quad (4.10)$$

and

$$H \frac{\partial V}{\partial t} \frac{\partial}{\partial r} \left( \frac{\partial t}{\partial r_1} \right) \frac{\partial r}{\partial r_1} + H \frac{\partial V}{\partial t} \frac{\partial}{\partial t} \left( \frac{\partial t}{\partial r_1} \right) \frac{\partial t}{\partial r_1} = H \frac{\partial V}{\partial t} \frac{\partial^2 t}{\partial r_1^2}. \quad (4.11)$$

Substituting (4.8) - (4.11) into (4.7) and then substituting  $z = H(r, t)V$  and the expressions for  $\frac{\partial z}{\partial r_2}$  and  $\frac{\partial^2 z}{\partial r_1^2}$  (given in (4.6) and (4.7) respectively) into (4.5) we get

$$\begin{aligned} & \left[ \frac{\partial V}{\partial r} \frac{\partial H}{\partial r} + V \frac{\partial^2 H}{\partial r^2} \right] \left( \frac{\partial r}{\partial r_1} \right)^2 + \left[ \frac{\partial H}{\partial r} \frac{\partial V}{\partial r} + H \frac{\partial^2 V}{\partial r^2} \right] \left( \frac{\partial r}{\partial r_1} \right)^2 + \\ & + \left[ \frac{\partial V}{\partial r} \frac{\partial H}{\partial t} + V \frac{\partial^2 H}{\partial r \partial t} \right] \frac{\partial t}{\partial r_1} \frac{\partial r}{\partial r_1} + \left[ \frac{\partial H}{\partial r} \frac{\partial V}{\partial t} + H \frac{\partial^2 V}{\partial r \partial t} \right] \frac{\partial t}{\partial r_1} \frac{\partial r}{\partial r_1} + \\ & + \left[ \frac{\partial V}{\partial t} \frac{\partial H}{\partial r} + V \frac{\partial^2 H}{\partial r \partial t} \right] \frac{\partial t}{\partial r_1} \frac{\partial r}{\partial r_1} + \left[ \frac{\partial H}{\partial t} \frac{\partial V}{\partial r} + H \frac{\partial^2 V}{\partial r \partial t} \right] \frac{\partial t}{\partial r_1} \frac{\partial r}{\partial r_1} + \\ & + \left[ \frac{\partial V}{\partial t} \frac{\partial H}{\partial t} + V \frac{\partial^2 H}{\partial t^2} \right] \left( \frac{\partial t}{\partial r_1} \right)^2 + \left[ \frac{\partial H}{\partial t} \frac{\partial V}{\partial t} + H \frac{\partial^2 V}{\partial t^2} \right] \left( \frac{\partial t}{\partial r_1} \right)^2 + \\ & + V \frac{\partial H}{\partial r} \frac{\partial^2 r}{\partial r_1^2} + H \frac{\partial V}{\partial r} \frac{\partial^2 r}{\partial r_1^2} + V \frac{\partial H}{\partial t} \frac{\partial^2 t}{\partial r_1^2} + H \frac{\partial V}{\partial t} \frac{\partial^2 t}{\partial r_1^2} + \\ & + V \frac{\partial H}{\partial r} \frac{\partial r}{\partial r_2} + H \frac{\partial V}{\partial r} \frac{\partial r}{\partial r_2} + V \frac{\partial H}{\partial t} \frac{\partial t}{\partial r_2} + H \frac{\partial V}{\partial t} \frac{\partial t}{\partial r_2} + \\ & + Q(r_1, r_2)H(r, t)V = 0. \end{aligned}$$

Simplifying we get

$$\begin{aligned} & \left[ V \frac{\partial^2 H}{\partial r^2} + 2 \frac{\partial V}{\partial r} \frac{\partial H}{\partial r} + H \frac{\partial^2 V}{\partial r^2} \right] \left( \frac{\partial r}{\partial r_1} \right)^2 + \\ & + \left[ 2 \frac{\partial V}{\partial r} \frac{\partial H}{\partial t} + 2 \frac{\partial H}{\partial r} \frac{\partial V}{\partial t} + 2V \frac{\partial^2 H}{\partial r \partial t} + 2H \frac{\partial^2 V}{\partial r \partial t} \right] \frac{\partial t}{\partial r_1} \frac{\partial r}{\partial r_1} + \\ & + \left[ V \frac{\partial^2 H}{\partial t^2} + 2 \frac{\partial V}{\partial t} \frac{\partial H}{\partial t} + H \frac{\partial^2 V}{\partial t^2} \right] \left( \frac{\partial t}{\partial r_1} \right)^2 + \left[ V \frac{\partial H}{\partial r} + H \frac{\partial V}{\partial r} \right] \left[ \frac{\partial^2 r}{\partial r_1^2} + \frac{\partial r}{\partial r_2} \right] + \\ & + \left[ V \frac{\partial H}{\partial t} + H \frac{\partial V}{\partial t} \right] \left[ \frac{\partial^2 t}{\partial r_1^2} + \frac{\partial t}{\partial r_2} \right] + Q(r_1, r_2)HV = 0. \quad (4.12) \end{aligned}$$

We want to equate the coefficients of the derivative terms in equation (4.12) with the coefficients of the derivative terms in equation (4.4). Looking at (4.4) we need the coefficient of  $\frac{\partial^2 V}{\partial r^2}$  to be 1. Hence dividing (4.12) throughout by

$$H \left( \frac{\partial r}{\partial r_1} \right)^2$$

we get

$$\begin{aligned} & \frac{V}{H} \frac{\partial^2 H}{\partial r^2} + \frac{2}{H} \frac{\partial V}{\partial r} \frac{\partial H}{\partial r} + \frac{\partial^2 V}{\partial r^2} + \\ & + \left[ 2 \frac{\partial V}{\partial r} \frac{\partial H}{\partial t} + 2 \frac{\partial H}{\partial r} \frac{\partial V}{\partial t} + 2V \frac{\partial^2 H}{\partial r \partial t} + 2H \frac{\partial^2 V}{\partial r \partial t} \right] \frac{(\partial t / \partial r_1)}{H(\partial r / \partial r_1)} + \\ & + \left[ V \frac{\partial^2 H}{\partial t^2} + 2 \frac{\partial V}{\partial t} \frac{\partial H}{\partial t} + H \frac{\partial^2 V}{\partial t^2} \right] \frac{(\partial t / \partial r_1)^2}{H(\partial r / \partial r_1)^2} + \\ & + \left[ V \frac{\partial H}{\partial r} + H \frac{\partial V}{\partial r} \right] \left[ \frac{\partial^2 r}{\partial r_1^2} + \frac{\partial r}{\partial r_2} \right] \frac{1}{H(\partial r / \partial r_1)^2} + \\ & + \left[ V \frac{\partial H}{\partial t} + H \frac{\partial V}{\partial t} \right] \left[ \frac{\partial^2 t}{\partial r_1^2} + \frac{\partial t}{\partial r_2} \right] \frac{1}{H(\partial r / \partial r_1)^2} + \\ & + \frac{1}{(\partial r / \partial r_1)^2} Q(r_1, r_2) V = 0. \end{aligned} \quad (4.13)$$

Now we look at (4.4) and compare coefficients of the remaining derivative terms with those of equation (4.13). We have ensured that the coefficient of  $\frac{\partial^2 V}{\partial r^2}$  is 1, so comparing the coefficient of  $\frac{\partial V}{\partial r}$  in (4.4) and (4.13) we get

$$\frac{2}{H} \frac{\partial H}{\partial r} + \frac{2}{H} \frac{\partial H}{\partial t} \frac{(\partial t / \partial r_1)}{(\partial r / \partial r_1)} + \frac{1}{(\partial r / \partial r_1)^2} \left[ \frac{\partial^2 r}{\partial r_1^2} + \frac{\partial r}{\partial r_2} \right] = 2 \left( \frac{u - \lambda w}{w^2} \right). \quad (4.14)$$

Comparing the coefficient of  $\frac{\partial V}{\partial t}$  in (4.4) and (4.13) we get

$$\frac{2}{H} \frac{\partial H}{\partial r} \frac{(\partial t / \partial r_1)}{(\partial r / \partial r_1)} + \frac{2}{H} \frac{\partial H}{\partial t} \frac{(\partial t / \partial r_1)^2}{(\partial r / \partial r_1)^2} + \frac{1}{(\partial r / \partial r_1)^2} \left[ \frac{\partial^2 t}{\partial r_1^2} + \frac{\partial t}{\partial r_2} \right] = \frac{2}{w^2}. \quad (4.15)$$

Comparing the coefficient of  $V$  in (4.4) and (4.13) we get

$$\begin{aligned} & \frac{1}{H} \frac{\partial^2 H}{\partial r^2} + \frac{2}{H} \frac{\partial^2 H}{\partial r \partial t} \frac{(\partial t / \partial r_1)}{(\partial r / \partial r_1)} + \frac{1}{H} \frac{\partial^2 H}{\partial t^2} \frac{(\partial t / \partial r_1)^2}{(\partial r / \partial r_1)^2} + \\ & + \frac{1}{H(\partial r / \partial r_1)^2} \frac{\partial H}{\partial r} \left[ \frac{\partial^2 r}{\partial r_1^2} + \frac{\partial r}{\partial r_2} \right] + \frac{1}{H(\partial r / \partial r_1)^2} \frac{\partial H}{\partial t} \left[ \frac{\partial^2 t}{\partial r_1^2} + \frac{\partial t}{\partial r_2} \right] + \\ & + \frac{Q(r_1, r_2)}{(\partial r / \partial r_1)^2} = \frac{-2r}{w^2}. \end{aligned} \quad (4.16)$$

The remaining derivative terms are

$$\frac{\partial^2 V}{\partial r \partial t} \text{ and } \frac{\partial^2 V}{\partial t^2}$$

and as they are not equal to zero (as that would lead to trivial solutions) their coefficients must be equal to zero. Hence we get the equation

$$2 \left( \frac{\partial t / \partial r_1}{\partial r / \partial r_1} \right) = 0 \quad (4.17)$$

from the coefficient of  $\frac{\partial^2 V}{\partial r \partial t}$  and the equation

$$\frac{(\partial t / \partial r_1)^2}{(\partial r / \partial r_1)^2} = 0 \quad (4.18)$$

from the coefficient of  $\frac{\partial^2 V}{\partial t^2}$ .

Hence the equations that we have to satisfy are (4.14)-(4.18). From (4.17) we can see that as

$$\frac{2}{\partial r / \partial r_1} \neq 0 \text{ hence } \frac{\partial t}{\partial r_1} = 0 \Rightarrow t = t(r_2). \quad (4.19)$$

As  $t = t(r_2)$  we notice that equation (4.18) is automatically satisfied. We now have to satisfy equations (4.14)-(4.16). Substituting  $\frac{\partial t}{\partial r_1} = 0$  into (4.14)-(4.16) we get, after some simplification

$$\frac{2}{H} \frac{\partial H}{\partial r} + \frac{1}{(\partial r / \partial r_1)^2} \left[ \frac{\partial^2 r}{\partial r_1^2} + \frac{\partial r}{\partial r_2} \right] = 2 \left( \frac{u - \lambda w}{w^2} \right), \quad (4.20)$$

$$\frac{1}{(\partial r / \partial r_1)^2} \frac{\partial t}{\partial r_2} = \frac{2}{w^2}, \text{ and} \quad (4.21)$$

$$\begin{aligned} \frac{1}{H} \frac{\partial^2 H}{\partial r^2} + \frac{1}{H(\partial r / \partial r_1)^2} \frac{\partial H}{\partial r} \left[ \frac{\partial^2 r}{\partial r_1^2} + \frac{\partial r}{\partial r_2} \right] + \frac{1}{H(\partial r / \partial r_1)^2} \frac{\partial H}{\partial t} \frac{\partial t}{\partial r_2} + \\ + \frac{Q(r_1, r_2)}{(\partial r / \partial r_1)^2} = \frac{-2r}{w^2}. \end{aligned} \quad (4.22)$$

So far we have  $t = t(r_2)$  and  $r = r(r_1, r_2)$  and equations (4.20)-(4.22) that we still have to satisfy. We consider three possible cases (for reasons of tractability) as outlined in Table (4.1) and solve the system (4.20) - (4.22) in each of these cases.

Case I	Case II	Case III
$t = t(r_2)$	$t = t(r_2)$	$t = t(r_2)$
$r = r(r_1)$	$r = r(r_1)$	$r = r(r_1)$
$w = w(r)$	$w = w(t)$	$w = A_1(r)A_2(t)$

Table 4.1: Outlining the 3 Cases

**CASE I:** Noting that in this case  $t = t(r_2), r = r(r_1)$  and that  $w = w(r)$  we can rewrite equations (4.20) - (4.22), after some simplification, as

$$\frac{\partial H}{\partial r} + \frac{H}{2(\partial r / \partial r_1)^2} \frac{\partial^2 r}{\partial r_1^2} = H \left( \frac{u - \lambda w}{w^2} \right), \quad (4.23)$$

$$\frac{\partial t}{\partial r_2} = \frac{2}{w^2} \left( \frac{\partial r}{\partial r_1} \right)^2 = m, \quad (4.24)$$

where  $m$  is an arbitrary constant and

$$\frac{\partial^2 H}{\partial r^2} + \frac{1}{(\partial r / \partial r_1)^2} \frac{\partial H}{\partial r} \frac{\partial^2 r}{\partial r_1^2} + \frac{1}{(\partial r / \partial r_1)^2} \frac{\partial H}{\partial t} \frac{\partial t}{\partial r_2} + \frac{Q(r_1, r_2)H}{(\partial r / \partial r_1)^2} + \frac{2rH}{w^2} = 0. \quad (4.25)$$

As  $t = t(r_2)$  and  $r = r(r_1)$  from (4.24) we get

$$\frac{dt}{dr_2} = m \Rightarrow t = mr_2 + C_1, \quad (4.26)$$

where  $C_1$  is an arbitrary constant and that

$$\left( \frac{dr}{dr_1} \right)^2 = w^2 \frac{m}{2},$$

so that

$$\int \frac{1}{w(r)} dr = \pm \sqrt{\frac{m}{2}} r_1 + C_2, \quad (4.27)$$

where  $C_2$  is an arbitrary constant. Also note that

$$\frac{\partial^2 r}{\partial r_1^2} = \frac{d}{dr_1} \left( \frac{dr}{dr_1} \right) = \sqrt{\frac{m}{2}} \frac{dw}{dr} \frac{dr}{dr_1} = \frac{m}{2} w \frac{dw}{dr}. \quad (4.28)$$

Hence equation (4.24) is satisfied with the solutions (to the two resulting ordinary differential equations) given in (4.26) and (4.27). We still have to satisfy equations (4.23) and (4.25). Substituting the expressions for  $\left(\frac{\partial r}{\partial r_1}\right)^2$  and  $\frac{\partial^2 r}{\partial r_1^2}$  into (4.23) we get

$$\frac{\partial H}{\partial r} + \frac{H}{2w} \frac{dw}{dr} = H \left( \frac{u - \lambda w}{w^2} \right).$$

Rearranging the terms slightly we get

$$\frac{1}{H} \frac{\partial H}{\partial r} = \left[ \frac{u - \lambda w}{w^2} - \frac{1}{2w} \frac{dw}{dr} \right],$$

and then integrating with respect to  $r$  we obtain

$$\ln H = \int \left[ \frac{u - \lambda w}{w^2} - \frac{1}{2w} \frac{dw}{dr} \right] dr + C_3(t),$$

finally giving us

$$H = C_4(t) w(r)^{-1/2} \exp \left( \int \frac{u - \lambda w}{w^2} dr \right), \quad (4.29)$$

where  $C_4$  is an arbitrary function of  $t$ .

With the value of  $H$  given in (4.29) equation (4.23) is satisfied. The last equation that we have to satisfy is (4.25). Note



$$\frac{\partial H}{\partial r} = H \left[ \frac{u - \lambda w}{w^2} - \frac{1}{2w} \frac{dw}{dr} \right]. \quad (4.30)$$

Also

$$\frac{\partial^2 H}{\partial r^2} = \frac{\partial H}{\partial r} \left[ \frac{u - \lambda w}{w^2} - \frac{1}{2w} \frac{dw}{dr} \right] + H \frac{\partial}{\partial r} \left[ \frac{u - \lambda w}{w^2} - \frac{1}{2w} \frac{dw}{dr} \right], \quad (4.31)$$

and

$$\frac{\partial H}{\partial t} = H \frac{\partial}{\partial t} \int \left( \frac{u - \lambda w}{w^2} - \frac{1}{2w} \frac{dw}{dr} \right) dr + \frac{C'_4(t)}{C_4(t)} H. \quad (4.32)$$

Substituting (4.30), (4.31), (4.32) and the expressions for  $\left(\frac{\partial r}{\partial r_1}\right)^2$  and  $\frac{\partial^2 r}{\partial r_1^2}$  into (4.25) then dividing throughout by  $H$  we get

$$\begin{aligned} & \left( \frac{u - \lambda w}{w^2} - \frac{1}{2w} \frac{dw}{dr} \right)^2 + \frac{\partial}{\partial r} \left( \frac{u - \lambda w}{w^2} - \frac{1}{2w} \frac{dw}{dr} \right) + \\ & + \frac{\partial(\ln w)}{\partial r} \left( \frac{u - \lambda w}{w^2} - \frac{1}{2w} \frac{dw}{dr} \right) + \frac{2}{w^2} \frac{\partial}{\partial t} \int \left( \frac{u - \lambda w}{w^2} - \frac{1}{2w} \frac{dw}{dr} \right) dr + \\ & + \frac{2}{w^2} \frac{C'_4(t)}{C_4(t)} + \frac{2}{w^2 m} Q(r_1, r_2) + \frac{2r}{w^2} = 0. \end{aligned}$$

Making  $Q(r_1, r_2)$  the subject in the above equation we get

$$\begin{aligned} Q(r_1, r_2) = -\frac{w^2 m}{2} \times & \left\{ \left( \frac{u - \lambda w}{w^2} - \frac{1}{2w} \frac{dw}{dr} \right)^2 + \frac{\partial}{\partial r} \left( \frac{u - \lambda w}{w^2} - \frac{1}{2w} \frac{dw}{dr} \right) + \right. \\ & + \frac{\partial(\ln w)}{\partial r} \left( \frac{u - \lambda w}{w^2} - \frac{1}{2w} \frac{dw}{dr} \right) + \frac{2}{w^2} \frac{\partial}{\partial t} \int \left( \frac{u - \lambda w}{w^2} - \frac{1}{2w} \frac{dw}{dr} \right) dr + \\ & \left. + \frac{2}{w^2} \frac{C'_4(t)}{C_4(t)} + \frac{2r}{w^2} \right\}. \quad (4.33) \end{aligned}$$

Therefore when  $t = t(r_2)$  (transformation given by (4.26)),  $r = r(r_1)$  (transformation given by (4.27), noting that  $w = w(r)$ ) and  $z = H(r, t)V$  (where  $H(r, t)$  is given by (4.29)) then equation (4.4) gets reduced to equation (4.5) where  $Q(r_1, r_2)$  is given by (4.33).

**CASE II:** Recall that we have to satisfy equations (4.20) - (4.22). Noting that in this case  $t = t(r_2)$ ,  $r = r(r_1)$  and  $w = w(t)$ , we can rewrite equations (4.20) - (4.22), after a little simplification, as

$$\frac{\partial H}{\partial r} + \frac{H}{2(\partial r / \partial r_1)^2} \frac{\partial^2 r}{\partial r_1^2} = H \left( \frac{u - \lambda w}{w^2} \right), \quad (4.34)$$

$$\frac{w^2}{2} \frac{\partial t}{\partial r_2} = \left( \frac{\partial r}{\partial r_1} \right)^2 = m, \quad (4.35)$$

where  $m$  is an arbitrary constant and

$$\frac{\partial^2 H}{\partial r^2} + \frac{1}{(\partial r / \partial r_1)^2} \frac{\partial H}{\partial r} \frac{\partial^2 r}{\partial r_1^2} + \frac{1}{(\partial r / \partial r_1)^2} \frac{\partial H}{\partial t} \frac{\partial t}{\partial r_2} + \frac{Q(r_1, r_2)H}{(\partial r / \partial r_1)^2} + \frac{2rH}{w^2} = 0. \quad (4.36)$$

Noting that  $t = t(r_2)$  and  $r = r(r_1)$ , from equation (4.35) we can solve

$$\frac{\partial t}{\partial r_2} \frac{w^2}{2} = m \Rightarrow \frac{dt}{dr_2} = 2mw^{-2} \Rightarrow \int w^2(t)dt = 2mr_2 + C_5 \quad (4.37)$$

where  $C_5$  is a constant of integration. Then (4.35) integrates to

$$r = \pm \sqrt{m} r_1 + C_6, \quad (4.38)$$

where  $C_6$  is a constant of integration. Note that

$$\frac{\partial^2 r}{\partial r_1^2} = \frac{d}{dr_1} \left( \frac{dr}{dr_1} \right) = \frac{d}{dr_1} (\sqrt{m}) = 0. \quad (4.39)$$

Hence equation (4.35) is satisfied with the solutions (to the resulting ordinary differential equations) given in (4.37) and (4.38). In a similar way in which we dealt with Case *I*, we now have to satisfy equations (4.34) and (4.36). Substituting the expressions for  $\left(\frac{\partial r}{\partial r_1}\right)^2$  and  $\frac{\partial^2 r}{\partial r_1^2}$  into (4.34) we get

$$\frac{\partial}{\partial r}(\ln H) = \left( \frac{u - \lambda w}{w^2} \right),$$

implying

$$H = C_7(t) \exp \int \left( \frac{u - \lambda w}{w^2} \right) dr, \quad (4.40)$$

where  $C_7$  is an arbitrary function of  $t$ .

With the value of  $H$  given in (4.40), equation (4.34) is satisfied leaving (4.36) to be satisfied. From (4.40) we have

$$\frac{\partial H}{\partial r} = H \left[ \frac{u - \lambda w}{w^2} \right]. \quad (4.41)$$

Note that

$$\frac{\partial^2 H}{\partial r^2} = \frac{\partial H}{\partial r} \left[ \frac{u - \lambda w}{w^2} \right] + H \frac{\partial}{\partial r} \left[ \frac{u - \lambda w}{w^2} \right],$$

which simplifies to

$$\frac{\partial^2 H}{\partial r^2} = H \left[ \frac{u - \lambda w}{w^2} \right]^2 + H \frac{\partial}{\partial r} \left[ \frac{u - \lambda w}{w^2} \right]. \quad (4.42)$$

Also

$$\begin{aligned} \frac{\partial H}{\partial t} = C_7(t) \exp \left( \int \left[ \frac{u - \lambda w}{w^2} \right] dr \right) \times \frac{\partial}{\partial t} \int \left[ \frac{u - \lambda w}{w^2} \right] dr + \\ + C_7'(t) \exp \left( \int \left[ \frac{u - \lambda w}{w^2} \right] dr \right), \end{aligned}$$

which simplifies to

$$\frac{\partial H}{\partial t} = H \frac{\partial}{\partial t} \int \left( \frac{u - \lambda w}{w^2} \right) dr + \frac{C_7'(t)}{C_7(t)} H. \quad (4.43)$$

Substituting (4.41), (4.42), (4.43) and the expressions for  $\left( \frac{\partial r}{\partial r_1} \right)^2$  and  $\frac{\partial^2 r}{\partial r_1^2}$  into (4.36)

and then dividing throughout by  $H$  we get

$$\begin{aligned} \left( \frac{u - \lambda w}{w^2} \right)^2 + \frac{\partial}{\partial r} \left( \frac{u - \lambda w}{w^2} \right) + \frac{2}{w^2} \left[ \frac{\partial}{\partial t} \int \left( \frac{u - \lambda w}{w^2} \right) dr + \frac{C_7'(t)}{C_7(t)} \right] + \\ + \frac{Q(r_1, r_2)}{m} + \frac{2r}{w^2} = 0. \end{aligned} \quad (4.44)$$

Hence making  $Q(r_1, r_2)$  the subject of (4.44) we get

$$\begin{aligned} Q(r_1, r_2) = -m \times \left\{ \left( \frac{u - \lambda w}{w^2} \right)^2 + \frac{\partial}{\partial r} \left( \frac{u - \lambda w}{w^2} \right) + \right. \\ \left. \frac{2}{w^2} \frac{\partial}{\partial t} \int \left( \frac{u - \lambda w}{w^2} \right) dr + \frac{2}{w^2} \frac{C_7'(t)}{C_7(t)} + \frac{2r}{w^2} \right\} \end{aligned} \quad (4.45)$$

Therefore when  $t = t(r_2)$  (transformation given by (4.37), noting that  $w$  is only a function of  $t$ ),  $r = r(r_1)$  (transformation given by (4.38) and  $z = H(r, t)V$  (where  $H(r, t)$  is given by (4.40)) then equation (4.4) is reduced to equation (4.5) where  $Q(r_1, r_2)$  is given by (4.45).

**Case III:** Noting that in this case  $t = t(r_2)$ ,  $r = r(r_1)$  and that  $w = A_1(r)A_2(t)$  we can rewrite equations (4.20) - (4.22), after a little simplification, as

$$\frac{\partial H}{\partial r} + \frac{H}{2(\partial r / \partial r_1)^2} \frac{\partial^2 r}{\partial r_1^2} = H \left( \frac{u - \lambda w}{w^2} \right), \quad (4.46)$$

$$\frac{\partial t}{\partial r_2} \frac{A_2^2(t)}{2} = \frac{1}{A_1^2(r)} \left( \frac{\partial r}{\partial r_1} \right)^2 = m, \quad (4.47)$$

where  $m$  is an arbitrary constant and

$$\frac{\partial^2 H}{\partial r^2} + \frac{1}{(\partial r / \partial r_1)^2} \frac{\partial H}{\partial r} \frac{\partial^2 r}{\partial r_1^2} + \frac{1}{(\partial r / \partial r_1)^2} \frac{\partial H}{\partial t} \frac{\partial t}{\partial r_2} + \frac{Q(r_1, r_2)H}{(\partial r / \partial r_1)^2} + \frac{2rH}{A_1^2(r)A_2^2(t)} = 0. \quad (4.48)$$

Noting that  $t = t(r_2)$  and  $r = r(r_1)$ , from equation (4.47) we get

$$\frac{\partial t}{\partial r_2} = 2A_2^{-2}(t)m \Rightarrow \int A_2^2(t)dt = 2mr_2 + C_8, \quad (4.49)$$

where  $C_8$  is a constant of integration. From (4.47) we also have to solve

$$\left( \frac{\partial r}{\partial r_1} \right)^2 = mA_1^2(r) \Rightarrow \frac{dr}{dr_1} = \pm A_1(r)\sqrt{m}, \quad (4.50)$$

yielding

$$\int \frac{dr}{A_1(r)} = \pm \sqrt{m} r_1 + C_9, \quad (4.51)$$

where  $C_9$  is a constant of integration. Note that

$$\frac{\partial^2 r}{\partial r_1^2} = \frac{d}{dr_1} \left( \frac{dr}{dr_1} \right) = \frac{d}{dr_1} (A_1(r) \sqrt{m}) = \frac{d}{dr} (A_1(r) \sqrt{m}) \frac{dr}{dr_1} = m A_1(r) \frac{dA_1(r)}{dr} \quad (4.52)$$

Hence by solving the two resulting ordinary differential equations from equation (4.47) (solutions given by (4.49) and (4.51) respectively) we have satisfied equation (4.47). Substituting the expressions for  $\left(\frac{\partial r}{\partial r_1}\right)^2$  and  $\frac{\partial^2 r}{\partial r_1^2}$  (given in (4.50) and (4.52) respectively) into (4.46) we get

$$\frac{\partial H}{\partial r} + \frac{H}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} = \left( \frac{u - \lambda w}{w^2} \right) H. \quad (4.53)$$

Simplifying (4.53) we get

$$\frac{\partial(\ln H)}{\partial r} = \left[ \frac{u - \lambda w}{w^2} - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right], \quad (4.54)$$

so that

$$H = C_{10}(t) \exp \int \left( \frac{u - \lambda w}{w^2} - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right) dr, \quad (4.55)$$

where  $C_{10}$  is an arbitrary function of  $t$ .

With this value of  $H$ , equation (4.46) is satisfied. The only equation left to be satisfied is (4.48). Note that

$$\frac{\partial H}{\partial r} = H \left( \frac{u - \lambda w}{w^2} - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right), \quad (4.56)$$

also

$$\begin{aligned} \frac{\partial^2 H}{\partial r^2} &= \frac{\partial H}{\partial r} \left( \frac{u - \lambda w}{w^2} - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right) + H \frac{\partial}{\partial r} \left( \frac{u - \lambda w}{w^2} - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right) \\ &= H \left( \frac{u - \lambda w}{w^2} - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right)^2 + H \frac{\partial}{\partial r} \left( \frac{u - \lambda w}{w^2} - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right), \end{aligned} \quad (4.57)$$

and lastly

$$\frac{\partial H}{\partial t} = H \frac{\partial}{\partial t} \int \left( \frac{u - \lambda w}{w^2} - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right) dr + \frac{C'_{10}(t)}{C_{10}(t)} H. \quad (4.58)$$

Now substituting (4.56), (4.57), (4.58) and the expressions for  $\left(\frac{\partial r}{\partial r_1}\right)^2$  and  $\frac{\partial^2 r}{\partial r_1^2}$  (given in (4.50) and (4.52) respectively) into (4.48) and then dividing throughout by  $H$  we get

$$\begin{aligned} &\left( \frac{u - \lambda w}{w^2} - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right)^2 + \frac{\partial}{\partial r} \left( \frac{u - \lambda w}{w^2} - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right) + \\ &\quad + \frac{1}{A_1(r)} \frac{\partial A_1(r)}{\partial r} \left( \frac{u - \lambda w}{w^2} - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right) + \\ &\quad + \frac{2}{A_1^2(r)A_2^2(t)} \left[ \frac{\partial}{\partial t} \int \left( \frac{u - \lambda w}{w^2} - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right) dr + \frac{C'_{10}(t)}{C_{10}(t)} \right] + \\ &\quad + \frac{Q(r_1, r_2)}{A_1^2(r)m} + \frac{2r}{w^2} = 0. \end{aligned} \quad (4.59)$$

Therefore making  $Q(r_1, r_2)$  the subject of (4.59) we get

$$\begin{aligned}
 Q(r_1, r_2) = & -mA_1^2(r) \times \left\{ \left( \frac{u - \lambda w}{w^2} - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right)^2 + \right. \\
 & + \frac{\partial}{\partial r} \left( \frac{u - \lambda w}{w^2} - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right) + \\
 & + \frac{1}{A_1(r)} \frac{\partial A_1(r)}{\partial r} \left( \frac{u - \lambda w}{w^2} - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right) + \\
 & \left. + \frac{2}{w^2} \frac{\partial}{\partial t} \int \left( \frac{u - \lambda w}{w^2} - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right) dr + \frac{2}{w^2} \frac{C'_{10}(t)}{C_{10}(t)} + \frac{2r}{w^2} \right\} \quad (4.60)
 \end{aligned}$$

Therefore when  $t = t(r_2)$  (transformation given by (4.49)),  $r = r(r_1)$  (transformation given by (4.51)) and  $z = H(r, t)$  (where  $H(r, t)$  is given by (4.55)) then equation (4.4) gets reduced to (4.5) where  $Q(r_1, r_2)$  is given by (4.60).

#### 4.1.1 Summary of results obtained

The Bond Pricing Equation (4.1) when rewritten as,

$$\frac{\partial^2 V}{\partial r^2} + 2b(r, t) \frac{\partial V}{\partial r} + \frac{2}{w^2} \frac{\partial V}{\partial t} - \frac{2r}{w^2} V = 0, \quad (4.61)$$

where

$$b(r, t) = \left( \frac{u - \lambda w}{w^2} \right),$$

can be reduced to

$$\frac{\partial^2 z}{\partial r_1^2} + \frac{\partial z}{\partial r_2} + Q(r_1, r_2)z = 0. \quad (4.62)$$



In the Case  $\mathbf{w} = \mathbf{w}(\mathbf{r})$ , the required transformation is

$$\begin{aligned}
 t &= mr_2 + a_1 \\
 \int \frac{1}{w(r)} dr &= \sqrt{\frac{m}{2}} r_1 + a_2 \\
 z &= H(r, t)V, \text{ where} \\
 H(r, t) &= w^{-1/2}c(t) \exp \int b(r, t)dr
 \end{aligned} \tag{4.63}$$

and where  $a_1$ ,  $a_2$  and  $m(\neq 0)$  are arbitrary constants and  $c(t)$  is an arbitrary function of  $t$ . Then  $Q(r_1, r_2)$  in (4.62) is given by

$$\begin{aligned}
 Q(r_1, r_2) &= -mr - \frac{mww'}{2} \left( b(r, t) - \frac{w'}{2w} \right) \\
 &\quad - \frac{w^2m}{2} \left[ \frac{\partial b}{\partial r} - \frac{1}{2} \left( \frac{ww'' - (w')^2}{w^2} \right) + \left( b(r, t) - \frac{w'}{2w} \right)^2 \right] \\
 &\quad - m \left( \frac{c'(t)}{c(t)} + \frac{\partial}{\partial t} \int b(r, t)dr \right).
 \end{aligned} \tag{4.64}$$

In the Case  $\mathbf{w} = \mathbf{w}(\mathbf{t})$ , the required transformation is

$$\begin{aligned}
 \int w^2(t)dt &= 2mr_2 + a_1 \\
 r &= \pm \sqrt{m}r_1 + a_2 \\
 z &= H(r, t)V, \text{ where} \\
 H(r, t) &= c(t) \exp \int b(r, t)dr
 \end{aligned} \tag{4.65}$$

and where  $a_1$ ,  $a_2$  and  $m(\neq 0)$  are arbitrary constants and  $c(t)$  is an arbitrary function of  $t$ . Then  $Q(r_1, r_2)$  in (4.62) is given by

$$Q(r_1, r_2) = -m \times \left\{ b^2(r, t) + \frac{\partial b}{\partial r} + \frac{2}{w^2} \frac{\partial}{\partial t} \int b(r, t) dr + \frac{2}{w^2} \frac{c'(t)}{c(t)} + \frac{2r}{w^2} \right\} \quad (4.66)$$

In the Case  $\mathbf{w} = \mathbf{A}_1(\mathbf{r})\mathbf{A}_2(\mathbf{t})$ , the required transformation is

$$\begin{aligned} \int A_2^2(t) dt &= 2mr_2 + a_1 \\ \int \frac{dr}{A_1(r)} &= \pm \sqrt{m}r_1 + a_2 \\ z &= H(r, t)V, \text{ where} \\ H(r, t) &= c(t) \exp \int \left( b(r, t) - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right) dr \end{aligned} \quad (4.67)$$

and where  $a_1$ ,  $a_2$  and  $m(\neq 0)$  are arbitrary constants and  $c(t)$  is an arbitrary function of  $t$ . Then  $Q(r_1, r_2)$  in (4.62) is given by

$$\begin{aligned} Q(r_1, r_2) &= -mA_1^2(r) \times \left\{ \left( b(r, t) - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right)^2 + \right. \\ &\quad \left. + \frac{\partial}{\partial r} \left( b(r, t) - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right) + \right. \\ &\quad \left. + \frac{1}{A_1(r)} \frac{\partial A_1(r)}{\partial r} \left( b(r, t) - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right) + \right. \\ &\quad \left. + \frac{2}{w^2} \left( \frac{c'(t)}{c(t)} + r + \frac{\partial}{\partial t} \int \left( b(r, t) - \frac{1}{2A_1(r)} \frac{\partial A_1(r)}{\partial r} \right) dr \right) \right\} \end{aligned} \quad (4.68)$$

As the case  $w = w(t)$  typically leads to negative interest rates, we choose to ignore that case here. The much more difficult case  $w = A_1(r)A_2(t)$  will be the subject of further investigation in the future. In this thesis we aim to find solutions

to (4.61) for the case  $w = w(r)$ . Hence we solve (4.62) with  $Q(r_1, r_2)$  given by (4.64) thus leading to a solution to the Bond Pricing Equation (4.61) by the relation between  $z$  and  $V$  given by (4.63). We do so by the method of Separation of Variables and by the method of Laplace Transforms.

## 4.2 Solving equation (4.62): Preliminary simplification

In order to solve equation (4.62) by Separation of Variables or by Laplace transforms we choose  $Q$  to be of the form

$$Q(r_1, r_2) = F(r_1) + G(r_2).$$

For simplicity we let  $m = 2$  in (4.64). We thus require

$$Q(r_1, r_2) = -2r + \frac{ww''}{2} - \frac{(w')^2}{4} - 2\frac{c'(t)}{c(t)} - \zeta, \quad (4.69)$$

where  $w = w(r)$ , and  $b(r, t)$  to be a solution to

$$w^2 \frac{\partial b}{\partial r} + w^2 b^2 + 2 \frac{\partial}{\partial t} \int b(r, t) dr = \zeta, \quad (4.70)$$

where  $\zeta$  can either be a constant, a function of  $r$ , a function of  $t$  or a linear combination of a function of  $r$  and a function of  $t$ . For now we let  $\zeta = 0$ <sup>1</sup> and will consider

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<sup>1</sup>In the case  $b = e^{pt} - r$ , so that the interest rate model displays mean-reversion, then  $\zeta = 0$

would imply that when  $t = 0$ , for small  $r$ , we would have  $w \approx p(1 + \frac{r}{2})$ .

more general forms of  $\zeta$  in Section (4.2.3). Differentiating (4.70) with respect to  $r$  we get

$$\frac{\partial}{\partial r} \left( \frac{w^2}{2} \left[ \frac{\partial b}{\partial r} + b^2 \right] \right) + \frac{\partial b}{\partial t} = 0. \quad (4.71)$$

Now we perform a change of variable by letting

$$b(r, t) = B^{-1} \frac{\partial B}{\partial r}$$

so that (4.71) becomes

$$\frac{\partial}{\partial r} \left( -\frac{w^2}{2} \frac{B_{rr}}{B} \right) = \frac{\partial}{\partial t} \left( \frac{\partial}{\partial r} \ln B \right),$$

which on integrating with respect to  $r$  and using

$$\int B^{-1} \frac{\partial B}{\partial r} dr = \ln B + \int g(t) dt$$

becomes

$$-\frac{w^2}{2} B_{rr} = B_t + g(t)B. \quad (4.72)$$

Now letting

$$B(r, t) = k(t)U(r, t),$$

where

$$k(t) = \exp \left( - \int g(t) dt \right),$$

equation (4.72) simplifies to

$$\frac{\partial^2 U}{\partial r^2} + \frac{2}{w(r)^2} \frac{\partial U}{\partial t} = 0. \quad (4.73)$$

From (4.63) we have

$$\begin{aligned} r_1 &= \int \frac{1}{w(r)} dr \\ t &= 2r_2 \end{aligned} \quad (4.74)$$

where we have set  $a_1 = a_2 = 0$ ,  $m = 2$ , and

$$\begin{aligned} V(r, t) &= \frac{z}{H(r, t)} \\ &= \frac{(w(r))^{1/2} z}{c(t) e^{\int b(r, t) dr}} \\ &= \frac{(w(r))^{1/2} z}{c(t) B(r, t) e^{\int g(t) dt}} \\ &= \frac{(w(r))^{1/2} z}{c(t) U(r, t)}. \end{aligned} \quad (4.75)$$

As

$$b = B^{-1} \frac{\partial B}{\partial r} = U^{-1} \frac{\partial U}{\partial r},$$

then without loss of generality we let  $g(t) = 0$  and hence  $B$  satisfies

$$\frac{\partial^2 B}{\partial r^2} + \frac{2}{w(r)^2} \frac{\partial B}{\partial t} = 0. \quad (4.76)$$

From (4.75) then

$$V(r, t) = \frac{(w(r))^{1/2} z}{c(t)B(r, t)}, \quad (4.77)$$

which needs to satisfy the final condition  $V(r, T) = 1$ , where  $T$  is the expiry time of the bond, so that

$$V(r, T) = 1 = \frac{(w(r))^{1/2}}{c(T)B(r, T)} z(r_1, T/2).$$

### 4.2.1 Solving equation (4.62) by Separation of Variables

In solving equation (4.62) by separation of variables we let

$$z = X(r_1)T(r_2), \quad (4.78)$$

and by substituting (4.78) into (4.62) we get

$$X''T + XT' + Q(r_1, r_2)XT = 0. \quad (4.79)$$

Substituting  $Q(r_1, r_2)$  from (4.69) into (4.79) and dividing by  $XT$  we get

$$\frac{X''}{X} + \frac{T'}{T} - 2r + \frac{ww''}{2} - \frac{(w')^2}{4} - 2\frac{c'(t)}{c(t)} = 0.$$

Rearranging the above equation we get

$$\frac{T'}{T} - 2\frac{c'(t)}{c(t)} = -\mu = -\frac{X''}{X} - \left(-2r + \frac{ww''}{2} - \frac{(w')^2}{4}\right) \quad (4.80)$$

where  $\mu$  is an arbitrary constant.

Solving the equation

$$\frac{T'}{T} - 2\frac{c'(t)}{c(t)} = -\mu$$

we get

$$T(r_2) = \beta c(2r_2)e^{-\mu r_2}, \quad (4.81)$$

where  $\beta$  is an arbitrary constant. The form of  $X(r)$  depends on our choice of  $w(r)$ , but from (4.80) needs to satisfy

$$\frac{X''}{X} - 2r + \frac{w''w}{2} - \frac{(w')^2}{4} = \mu. \quad (4.82)$$

Note that from (4.77) we have

$$V(r, t) = \frac{w(r)^{1/2}}{c(2r_2)B(r, t)}z(r_1, r_2),$$

where  $B(r, t)$  is a solution to (4.76) so that when equation (4.62) is solved by separation of variables, the final condition

$$V(r, T) = 1 = \frac{(w(r))^{1/2}}{B(r, T)}\beta e^{-\mu T/2}X(r_1),$$

means that we require

$$B(r, T) = \beta(w(r))^{1/2}e^{-\mu T/2}X(r_1). \quad (4.83)$$

In order to solve (4.76) by Laplace Transforms, it will be convenient to first make the substitution  $\bar{t} = T - t$  so that (4.76) becomes

$$\frac{\partial^2 B}{\partial r^2} - \frac{2}{w(r)^2} \frac{\partial B}{\partial \bar{t}} = 0, \quad (4.84)$$

which from (4.83) we solve subject to

$$B(r, 0) = \beta(w(r))^{1/2} e^{-\mu T/2} X(r_1). \quad (4.85)$$

We now consider two specific forms of  $w(r)$ .

**Example-1:**  $w(r) = r^2$ : Substituting  $w(r) = r^2$  into (4.82) we find that  $X$  needs to satisfy

$$X'' + (-\mu + 2/r_1)X = 0, \quad (4.86)$$

noting that from (4.74)

$$r_1 = -\frac{1}{r}.$$

By letting  $X = e^{\sqrt{\mu}r_1}u(r_1)$  and then  $u = v(x)$ ,  $x = -2\sqrt{\mu}r_1$ ; for  $\mu > 0$ , we find that  $v$  needs to satisfy

$$xv'' - xv' - \frac{v}{\sqrt{\mu}} = 0. \quad (4.87)$$

The general solution to (4.87) is

$$v = x \left[ b_1 \Phi \left( 1 + \frac{1}{\sqrt{\mu}}; 2; x \right) + b_2 \Psi \left( 1 + \frac{1}{\sqrt{\mu}}; 2; x \right) \right],$$



where  $b_1$  and  $b_2$  are arbitrary constants and  $\Phi$  and  $\Psi$  represent the Confluent Hypergeometric functions KummerM and KummerU respectively so that

$$X(r_1) = e^{\sqrt{\mu}r_1} r_1 \left[ c_1 \Phi \left( 1 + \frac{1}{\sqrt{\mu}}; 2; -2\sqrt{\mu}r_1 \right) + c_2 \Psi \left( 1 + \frac{1}{\sqrt{\mu}}; 2; -2\sqrt{\mu}r_1 \right) \right], \quad (4.88)$$

where  $c_1$  and  $c_2$  are arbitrary constants.

Hence a solution to equation (4.62) with  $w(r) = r^2$  is

$$\begin{aligned} z(r_1, r_2) &= X(r_1)T(r_2) \\ &= c(2r_2)e^{-\mu r_2} e^{\sqrt{\mu}r_1} r_1 \left[ c_1 \Phi \left( 1 + \frac{1}{\sqrt{\mu}}; 2; -2\sqrt{\mu}r_1 \right) \right. \\ &\quad \left. + c_2 \Psi \left( 1 + \frac{1}{\sqrt{\mu}}; 2; -2\sqrt{\mu}r_1 \right) \right] \quad \text{for } \mu > 0. \end{aligned} \quad (4.89)$$

From (4.84) and (4.85) we now need to solve

$$\frac{\partial^2 B}{\partial r^2} - \frac{2}{r^4} \frac{\partial B}{\partial \bar{t}} = 0, \quad (4.90)$$

subject to

$$\begin{aligned} B(r, 0) &= e^{-\mu T/2} e^{-\sqrt{\mu}/r} \left[ c_1 \Phi \left( 1 + \frac{1}{\sqrt{\mu}}; 2; \frac{2\sqrt{\mu}}{r} \right) + \right. \\ &\quad \left. + c_2 \Psi \left( 1 + \frac{1}{\sqrt{\mu}}; 2; \frac{2\sqrt{\mu}}{r} \right) \right] \quad \text{for } \mu > 0. \end{aligned} \quad (4.91)$$

Taking Laplace Transforms of (4.90) with respect to  $\bar{t}$ , we get

$$\frac{d^2}{dr^2} \tilde{B}(r, p) - \frac{2}{r^4} [p\tilde{B}(r, p) - B(r, 0)] = 0,$$

where  $\tilde{B}(r, p) = \mathcal{L}\{B(r, \bar{t})\}$ , which simplifies to

$$r^4 \frac{d^2 \tilde{B}}{dr^2} - 2p\tilde{B} = -2B(r, 0). \quad (4.92)$$

The solution to (4.92) may be written as  $\tilde{B} = \tilde{B}_c + \tilde{B}_p$  where  $\tilde{B}_c$  is the solution to the corresponding homogeneous equation of (4.92), i.e.

$$r^4 \frac{d^2 \tilde{B}_c}{dr^2} - 2p \tilde{B}_c = 0, \quad (4.93)$$

and  $\tilde{B}_p$  is a particular solution to (4.92). We first find  $\tilde{B}_c$ . Letting  $u = \tilde{B}_c/r$ ,  $z = 1/r$  in (4.93) we get

$$\frac{d^2 u}{dz^2} - 2pu = 0,$$

which is a constant coefficient second order ordinary differential equation, so that as  $p > 0$ ,

$$u = A(p)e^{z\sqrt{2p}} + D(p)e^{-z\sqrt{2p}},$$

giving us

$$\tilde{B}_c = r \left[ A(p)e^{\sqrt{2p}/r} + D(p)e^{-\sqrt{2p}/r} \right], \quad (4.94)$$

where  $\tilde{B}_c$  denotes the complementary solution to (4.92) and where  $A(p)$  and  $D(p)$  are arbitrary functions of the transform variable  $p$ .

Let us call  $\tilde{B}_c = r \left( \tilde{B}_{c_1} + \tilde{B}_{c_2} \right)$ , where  $\tilde{B}_{c_1} = A(p)e^{\sqrt{2p}/r}$  and  $\tilde{B}_{c_2} = D(p)e^{-\sqrt{2p}/r}$ . Then Table (4.2) outlines possible invertible choices for  $\tilde{B}_{c_2}$  (from the F(p) column) with their respective Inverse Laplace Transforms. Here,  $a = \sqrt{2}/r$ .

So any choice for  $\tilde{B}_{c_1}$  and  $\tilde{B}_{c_2}$  from a column in Table (4.2) will give us an analytic solution to  $\tilde{B}_c$ . We can get a total of 324 classes of solutions for the various choices of  $\tilde{B}_{c_1}$  and  $\tilde{B}_{c_2}$  as listed in Table (4.2).

As an example suppose we choose  $A(p) = 0$  and  $D(p) = c_4 + c_3 p^{-1/2}$ . Then

$$B_c = r \left[ \frac{c_3}{\sqrt{\pi \bar{t}}} e^{-\frac{1}{2r^2 \bar{t}}} + \frac{c_4}{r\sqrt{2\pi \bar{t}^{3/2}}} e^{-\frac{1}{2r^2 \bar{t}}} \right], \quad \text{where } \bar{t} = T - t. \quad (4.95)$$

$F(p)$	$\mathcal{L}^{-1}\{F(p)\}$
$c_1 e^{-ap^{\frac{1}{2}}}$	$\frac{1}{2}c_1 a \pi^{-\frac{1}{2}} \bar{t}^{-\frac{3}{2}} \exp(-a^2/(4\bar{t}))$
$c_1 p^{\frac{1}{2}} e^{-ap^{\frac{1}{2}}}$	$\frac{1}{4}c_1 \pi^{-\frac{1}{2}} (a^2 - 2\bar{t}) \bar{t}^{-\frac{5}{2}} \exp(-a^2/(4\bar{t}))$
$c_1 p^{-\frac{1}{2}} e^{-ap^{\frac{1}{2}}}$	$c_1 (\pi \bar{t})^{-\frac{1}{2}} \exp(-a^2/(4\bar{t}))$
$c_1 p e^{-ap^{\frac{1}{2}}}$	$\frac{1}{4}c_1 a \pi^{-\frac{1}{2}} \bar{t}^{-\frac{5}{2}} (a^2/(2\bar{t}) - 3) \exp(-a^2/(4\bar{t}))$
$c_1 p^{-1} e^{-ap^{\frac{1}{2}}}$	$c_1 \text{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}})$
$c_1 p^{\frac{3}{2}} e^{-ap^{\frac{1}{2}}}$	$\frac{1}{4}c_1 \pi^{-\frac{1}{2}} \bar{t}^{-\frac{5}{2}} (3 - 3a^2/(2\bar{t}) + a^4/(4\bar{t}^2))$
$c_1 p^{-\frac{3}{2}} e^{-ap^{\frac{1}{2}}}$	$2c_1 (\bar{t}/\pi)^{\frac{1}{2}} \exp(-a^2/(4\bar{t})) - a \text{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}})$
$c_1 p^{\frac{1}{2}n - \frac{1}{2}} e^{-(ap)^{\frac{1}{2}}}$	$c_1 (2\bar{t})^{-\frac{1}{2}n} (\pi \bar{t})^{-\frac{1}{2}} \exp(-\frac{1}{4}a/\bar{t}) \text{He}_n[(2\bar{t}/a)^{-\frac{1}{2}}]$ where $\text{He}_n(r)$ is the Hermite's polynomial of order $n$ defined as $\text{He}_n(r) = (-1)^n e^{\frac{1}{2}r^2} \frac{d^n}{dr^n} e^{-\frac{1}{2}r^2}$
$c_1 p^\nu e^{-ap^{\frac{1}{2}}}$	$2^{-\nu - \frac{1}{2}} c_1 \pi^{-\frac{1}{2}} \bar{t}^{-\nu - 1} \exp(-a^2/(8\bar{t})) \text{D}_{2\nu+1}[a(2\bar{t})^{-\frac{1}{2}}]$ where $\text{D}_\nu(r)$ is the parabolic cylindrical function defined as $\text{D}_\nu(r) = e^{-\frac{1}{4}r^2} \text{He}_\nu(r)$ , $\nu = 0, 1, 2, \dots$
$c_1 e^{-ap^{\frac{1}{2}}} (p^{\frac{1}{2}} + b)^{-1}$	$c_1 (\pi \bar{t})^{-\frac{1}{2}} \exp(-a^2/(4\bar{t})) - c_1 b \exp(ab + b^2 \bar{t}) \text{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}} + b\bar{t}^{\frac{1}{2}})$
$c_1 p^{-\frac{1}{2}} e^{-ap^{\frac{1}{2}}} (p^{\frac{1}{2}} + b)^{-1}$	$c_1 \exp(ab + b^2 \bar{t}) \text{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}} + b\bar{t}^{\frac{1}{2}})$
$c_1 p^{-1} (p^{\frac{1}{2}} + b)^{-1} e^{-ap^{\frac{1}{2}}}$	$c_1 b^{-1} \text{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}}) - c_1 b^{-1} \exp(ab + b^2 \bar{t}) \text{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}} + b\bar{t}^{\frac{1}{2}})$
$c_1 p^{\frac{1}{2}} (p^{\frac{1}{2}} + b)^{-1} e^{-ap^{\frac{1}{2}}}$	$c_1 (\pi \bar{t})^{-\frac{1}{2}} (\frac{1}{2}a/\bar{t} - b) \exp(-\frac{1}{4}a^2/\bar{t})$ $+ c_1 b^2 \exp(ab + b^2 \bar{t}) \text{Erfc}(\frac{1}{2}a\bar{t}^{-\frac{1}{2}} + b\bar{t}^{\frac{1}{2}})$
$\vdots$	$\vdots$

$\vdots$	$\vdots$
$c_1 p(p^{\frac{1}{2}} + b)^{-1} e^{-ap^{\frac{1}{2}}}$	$c_1 \pi^{-\frac{1}{2}} \bar{t}^{-\frac{3}{2}} (b^2 \bar{t} - \frac{1}{2} - \frac{1}{2} ab + \frac{1}{4} a^2 / \bar{t}) \exp(-\frac{1}{4} a^2 / \bar{t})$ $- c_1 b^3 \exp(ab + b^2 \bar{t}) \operatorname{Erfc}(\frac{1}{2} a \bar{t}^{-\frac{1}{2}} + b \bar{t}^{\frac{1}{2}})$
$c_1 p^{-\frac{3}{2}} (p^{\frac{1}{2}} + b)^{-1} e^{-ap^{\frac{1}{2}}}$	$c_1 b^{-1} [2(\bar{t}/\pi)^{\frac{1}{2}} \exp(-\frac{1}{4} a^2 / \bar{t}) - (a + b^{-1}) \operatorname{Erfc}(\frac{1}{2} a \bar{t}^{-\frac{1}{2}})$ $+ b^{-1} \exp(ab + b^2 \bar{t}) \operatorname{Erfc}(\frac{1}{2} a \bar{t}^{-\frac{1}{2}} + b \bar{t}^{\frac{1}{2}})]$
$c_1 (p^{\frac{1}{2}} + b)^{-2} e^{-ap^{\frac{1}{2}}}$	$c_1 (2b\bar{t}^2 + ab + 1) \exp(ab + b^2 \bar{t}) \operatorname{Erfc}(\frac{1}{2} a \bar{t}^{-\frac{1}{2}} + b \bar{t}^{\frac{1}{2}})$ $- 2c_1 b(\bar{t}/\pi)^{\frac{1}{2}} \exp(-\frac{1}{4} a^2 / \bar{t})$
$c_1 p^{-\frac{1}{2}} (p^{\frac{1}{2}} + b)^{-2} e^{-ap^{\frac{1}{2}}}$	$2c_1 (\bar{t}/\pi)^{\frac{1}{2}} \exp(-\frac{1}{4} a^2 / \bar{t})$ $- c_1 (2b\bar{t} + a) \exp(ab + b^2 \bar{t}) \operatorname{Erfc}(\frac{1}{2} a \bar{t}^{-\frac{1}{2}} + b \bar{t}^{\frac{1}{2}})$
$c_1 p^{-1} (p^{\frac{1}{2}} + b)^{-2} e^{-ap^{\frac{1}{2}}}$	$c_1 b^{-2} \operatorname{Erfc}(\frac{1}{2} a \bar{t}^{-\frac{1}{2}}) - 2c_1 b^{-1} (\bar{t}/\pi) \exp(-\frac{1}{4} a^2 / \bar{t})$ $+ c_1 (2\bar{t} - b^{-2} \frac{a}{b}) \exp(ab + b^2 \bar{t}) \operatorname{Erfc}(\frac{1}{2} a \bar{t}^{-\frac{1}{2}} + b \bar{t}^{\frac{1}{2}})$

Table 4.2: Outlining the possible solutions to the homogeneous part of (4.92)

To find  $\tilde{B}_p$ , a particular solution to (4.92) we use the result from variation of parameters that if a nonhomogeneous linear equation of the second order has the form

$$f_2(r)y_{rr} + f_1(r)y_r + f_0(r)y = g(r),$$

and if  $y_1 = y_1(r)$  and  $y_2 = y_2(r)$  be two non-trivial linearly-independent solutions of the corresponding homogeneous equation with  $g \neq 0$ , then the particular solution  $y_p$  of the above equation can be found from the formula

$$y_p = y_2 \int y_1 \frac{g}{f_2} \frac{dr}{W} - y_1 \int y_2 \frac{g}{f_2} \frac{dr}{W}, \quad (4.96)$$

where  $W$  is the Wronskian,  $W = y_1(y_2)_r - y_2(y_1)_r$ . Hence in order to find a particular solution to (4.92) we let

$$y_1 = rA(p)e^{\sqrt{2p}/r} \quad \text{and} \quad y_2 = rD(p)e^{-\sqrt{2p}/r},$$

so that  $W$  simplifies to

$$W = 2\sqrt{2p}A(p)D(p),$$

and

$$\frac{g}{f_2} = -\frac{2B(r, 0)}{r^4} = -\frac{2}{r^4}e^{-\mu T/2}e^{-\sqrt{\mu}/r} \left[ c_1\Phi\left(1 + \frac{1}{\sqrt{\mu}}; 2; \frac{2\sqrt{\mu}}{r}\right) + c_2\Psi\left(1 + \frac{1}{\sqrt{\mu}}; 2; \frac{2\sqrt{\mu}}{r}\right) \right].$$

Hence, after a little simplification we get

$$\begin{aligned} \tilde{B}_p(r, p) &= y_1 \int y_2 \frac{g}{f_2} \frac{dr}{W} - y_2 \int y_1 \frac{g}{f_2} \frac{dr}{W} \\ &= -\frac{re^{-\sqrt{2p}/r}e^{-\mu T/2}}{\sqrt{2p}} \int_0^r \frac{e^{(\sqrt{2p}-\sqrt{\mu})/r}}{r^3} \left[ c_3\Phi\left(1 + \frac{1}{\sqrt{\mu}}; 2; \frac{2\sqrt{\mu}}{r}\right) + c_4\Psi\left(1 + \frac{1}{\sqrt{\mu}}; 2; \frac{2\sqrt{\mu}}{r}\right) \right] dr \\ &\quad + \frac{re^{\sqrt{2p}/r}e^{-\mu T/2}}{\sqrt{2p}} \int_0^r \frac{e^{-(\sqrt{2p}+\sqrt{\mu})/r}}{r^3} \left[ c_3\Phi\left(1 + \frac{1}{\sqrt{\mu}}; 2; \frac{2\sqrt{\mu}}{r}\right) + c_4\Psi\left(1 + \frac{1}{\sqrt{\mu}}; 2; \frac{2\sqrt{\mu}}{r}\right) \right] dr. \end{aligned} \tag{4.97}$$

Now we need to find the Laplace inverse of  $\tilde{B}_p$ . For the sake of simplicity, let us define

$$h(r) = \frac{e^{-\sqrt{\mu}/r}}{r^3} \left[ c_3\Phi\left(1 + \frac{1}{\sqrt{\mu}}; 2; \frac{2\sqrt{\mu}}{r}\right) + c_4\Psi\left(1 + \frac{1}{\sqrt{\mu}}; 2; \frac{2\sqrt{\mu}}{r}\right) \right] \tag{4.98}$$

and also rewrite  $\tilde{B}_p$  as

$$\tilde{B}_p = re^{-\mu T/2}(\tilde{B}_{p_1} + \tilde{B}_{p_2}), \tag{4.99}$$

where

$$\tilde{B}_{p_2} = -\frac{e^{-\sqrt{2p}/r}}{\sqrt{2p}} \int_0^r e^{\sqrt{2p}/r} h(r) dr,$$

$$\tilde{B}_{p_1} = \frac{e^{\sqrt{2p}/r}}{\sqrt{2p}} \int_0^r e^{-\sqrt{2p}/r} h(r) dr.$$

If we now let

$$B_e(r, \bar{t}) = \mathcal{L}^{-1} \left\{ \frac{e^{\frac{\sqrt{2p}}{r}}}{\sqrt{2p}} \int_0^r e^{-\frac{\sqrt{2p}}{r_3}} h_e(r_3) dr_3 \right\}, \text{ and} \quad (4.100)$$

$$B_o(r, \bar{t}) = \mathcal{L}^{-1} \left\{ \frac{e^{\frac{\sqrt{2p}}{r}}}{\sqrt{2p}} \int_0^r e^{-\frac{\sqrt{2p}}{r_3}} h_o(r_3) dr_3 \right\}$$

where

$$h_e = \frac{h(r) + h(-r)}{2}, \text{ the even part of } h, \text{ and}$$

$$h_o = \frac{h(r) - h(-r)}{2}, \text{ the odd part of } h,$$

then

$$B_p(r, \bar{t}) = r e^{\frac{-\mu T}{2}} \{B_e(r, \bar{t}) + B_o(r, \bar{t}) + B_e(-r, \bar{t}) - B_o(-r, \bar{t})\}. \quad (4.101)$$

We note firstly that

$$\begin{aligned} \mathcal{L}^{-1} \left\{ e^{\frac{p}{r}} \int_0^r e^{\frac{-p}{r_3}} h_e(r_3) dr_3 \right\} &= \int_0^r \mathcal{L}^{-1} \left( e^{-p \left( \frac{1}{r_3} - \frac{1}{r} \right)} h_e(r_3) \right) dr_3 \\ &= \int_0^r \delta \left( \bar{t} - \left( \frac{1}{r_3} - \frac{1}{r} \right) \right) h_e(r_3) dr_3. \end{aligned}$$

With the substitution  $r_4 = -\frac{1}{r_3}$ , we get

$$\begin{aligned} \int_0^r \delta \left( \bar{t} - \left( \frac{1}{r_3} - \frac{1}{r} \right) \right) h_e(r_3) dr_3 &= \int_{-\infty}^{-\frac{1}{r}} \delta \left( \bar{t} + r_4 + \frac{1}{r} \right) h_e \left( -\frac{1}{r_4} \right) \frac{1}{r_4^2} dr_4 \\ &= h_e \left( \left[ \bar{t} + \frac{1}{r} \right]^{-1} \right) \left( \frac{1}{\bar{t} + \frac{1}{r}} \right)^2. \end{aligned}$$

Hence

$$\mathcal{L}^{-1} \left\{ \frac{e^{\frac{\sqrt{p}}{r}}}{\sqrt{p}} \int_0^r e^{\frac{-\sqrt{p}}{r_3}} h_e(r_3) dr_3 \right\} = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{\frac{-u^2}{4t}} h_e \left( \left[ u + \frac{1}{r} \right]^{-1} \right) \left( \frac{1}{u + \frac{1}{r}} \right)^2 du$$

so that

$$B_e(r, \bar{t}) = \frac{1}{\sqrt{2\pi \bar{t}}} \int_0^\infty e^{\frac{-u^2}{2\bar{t}}} h_e \left( \left[ u + \frac{1}{r} \right]^{-1} \right) \left( \frac{1}{u + \frac{1}{r}} \right)^2 du.$$

Similarly it can be shown that

$$B_o(r, \bar{t}) = \frac{1}{\sqrt{2\pi \bar{t}}} \int_0^\infty e^{\frac{-u^2}{2\bar{t}}} h_o \left( \left[ u + \frac{1}{r} \right]^{-1} \right) \left( \frac{1}{u + \frac{1}{r}} \right)^2 du.$$

Hence we have from (4.101) that

$$\begin{aligned} B_p(r, \bar{t}) &= r e^{-\frac{\mu T}{2}} \left\{ \frac{1}{\sqrt{2\pi \bar{t}}} \int_0^\infty e^{\frac{-u^2}{2\bar{t}}} h \left( \left[ u + \frac{1}{r} \right]^{-1} \right) \left( \frac{1}{u + \frac{1}{r}} \right)^2 du \right. \\ &\quad + \frac{1}{\sqrt{2\pi \bar{t}}} \int_0^\infty e^{\frac{-u^2}{2\bar{t}}} h \left( \left[ u - \frac{1}{r} \right]^{-1} \right) \left( \frac{1}{u - \frac{1}{r}} \right)^2 du \\ &\quad \left. - \frac{\sqrt{2}}{\sqrt{\pi \bar{t}}} \int_0^\infty e^{\frac{-u^2}{2\bar{t}}} h_o \left( \left[ u - \frac{1}{r} \right]^{-1} \right) \left( \frac{1}{u - \frac{1}{r}} \right)^2 du \right\}. \end{aligned} \quad (4.102)$$

Finally, a solution to (4.90), (4.91) is

$$B(r, \bar{t}) = B_c(r, \bar{t}) + B_p(r, \bar{t}). \quad (4.103)$$

where  $B_c = \mathcal{L}^{-1}\{\tilde{B}_c\}$  which can be found from (4.94), and Table (4.2), (an example is given in (4.95)) and  $B_p$  is given in (4.102).

We have effectively finished one case and now collect the results for the final solution. From (4.77) and (4.89) a solution of the BPE ((4.61) with the final condition  $V(r, T) = 1$ ) is

$$V(r, t) = \frac{r}{c(t)B(r, t)} z\left(-\frac{1}{r}, \frac{t}{2}\right),$$

where

$$z(r_1, r_2) = c(2r_2)e^{-\mu r_2}e^{\sqrt{\mu}r_1}r_1 \left[ c_1\Phi\left(1 + \frac{1}{\sqrt{\mu}}; 2; -2\sqrt{\mu}r_1\right) + c_2\Psi\left(1 + \frac{1}{\sqrt{\mu}}; 2; -2\sqrt{\mu}r_1\right) \right],$$

$r_1 = -\frac{1}{r}$ ,  $r_2 = \frac{t}{2}$  and where  $B(r, \bar{t})$  is given by (4.103) noting that  $\bar{t} = T - t$ .

The stochastic differential equation for the spot risk-neutral rate in this case (when  $w(r) = r^2$ ) takes the form

$$dr = r^4 b(r, t)dt + r^2 dX,$$

where  $b(r, t) = \frac{u - \lambda w}{w^2}$  whose form we can get from the solution of  $B(r, t)$  (given in equation (4.103)) by the transformation  $b = B^{-1} \frac{\partial B}{\partial r}$ .

We note that with 324 different solutions for  $B$ , we can obtain for this  $w(r) = r^2$  example many possible forms for the stochastic differential equation for  $r$ . Recall that for interest rate derivative products, the yield curve provides one way to get a measure of how much the product ‘earns’. Figure (4.1) shows a sample solution of the BPE and Figure (4.2) shows the corresponding yield curve.



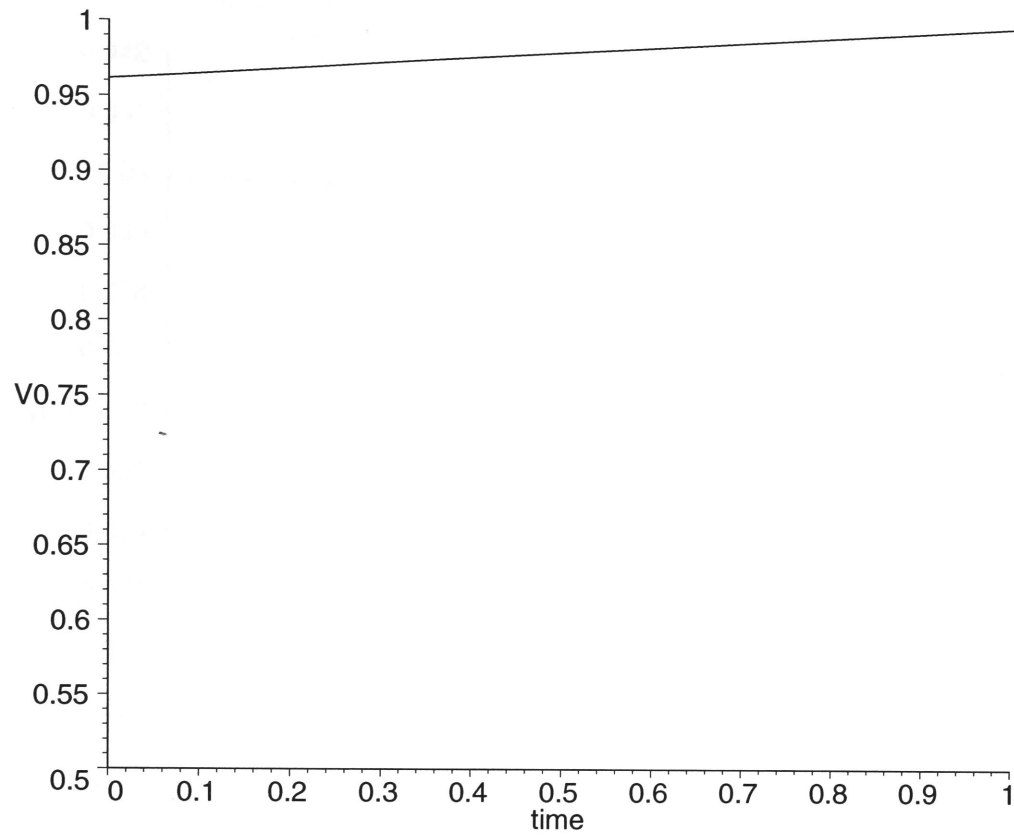


Figure 4.1: Sample solution for the BPE. Parameter values:  $r = 0.04, T = 1$ .

Constants preceding  $\Psi(\cdot)$  were 0 and all other constants were taken to be unity.

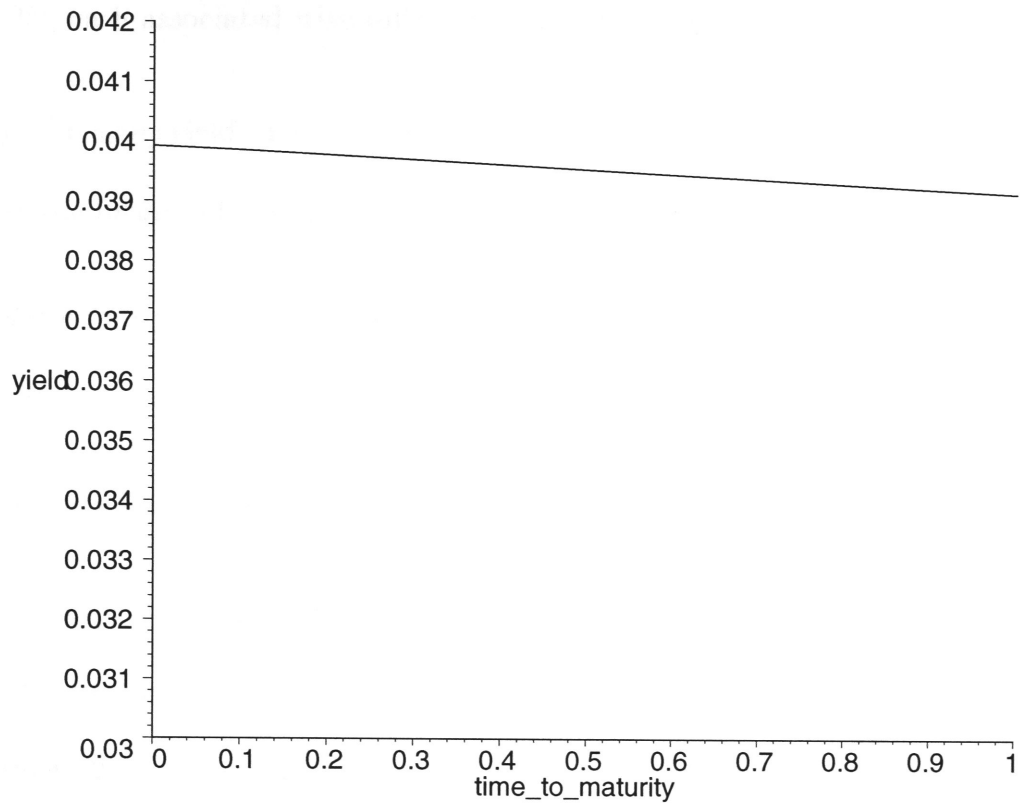


Figure 4.2: Sample yield curve for the BPE. Parameter values:  $r = 0.04, T = 1$ .

Constants preceding  $\Psi(\cdot)$  were 0 and all other constants were taken to be unity.

There are three commonly found yield curve shapes that are observed in the market [25], each associated with different economic conditions:

- An *increasing* yield curve is the most common form for the curve. This implies that the future interest rates are higher than the short-term rate.
- A *decreasing* yield curve is observed when the short rate is high but is expected to fall.
- A *humped* yield curve predicts a fall in the short rate.

In Figure (4.2) we showed a sample yield curve that maybe obtained where  $w = r^2$ . We now proceed to get analytical solutions for another example.

**Example-2:**  $w(r) = -2r^{3/2}$  (or equally we could take  $w(r) = 2r^{3/2}$ ): In this case from (4.69)  $Q(r_1, r_2)$  becomes

$$\begin{aligned} Q(r_1, r_2) &= F(r_1) + G(r_2), \\ &= -2r + \frac{ww''}{2} - \frac{(w')^2}{4} - 2\frac{c'(t)}{c(t)}, \\ &= -\frac{11}{4}r - 2\frac{c'(t)}{c(t)}. \end{aligned} \tag{4.104}$$

From (4.81), we know that for

$$z = X(r_1)T(r_2),$$

$$T(r_2) = \beta c(2r_2)e^{-\mu r_2}.$$

From (4.82) the corresponding  $X$  equation with  $w(r) = -2r^{3/2}$  is

$$X'' + \left[ -\frac{11}{4} \frac{1}{r_1^2} - \mu \right] X = 0. \tag{4.105}$$

where from (4.74)  $r_1 = \frac{1}{\sqrt{r}}$ . Equation (4.105) has a solution given by [21],

$$X(r_1) = r_1^{1/2} \left[ c_1 J_{\sqrt{3}}(\sqrt{-\mu}r_1) + c_2 J_{-\sqrt{3}}(\sqrt{-\mu}r_1) \right], \mu < 0$$

where  $J_\nu$  is the Bessel function of the first kind. Hence a solution to equation (4.62)

with  $w(r) = -2r^{3/2}$  is

$$\begin{aligned} z(r_1, r_2) &= X(r_1)T(r_2) \\ &= \beta c(2r_2)e^{-\mu r_2} r_1^{1/2} \left[ c_1 J_{\sqrt{3}}(\sqrt{-\mu}r_1) + c_2 J_{-\sqrt{3}}(\sqrt{-\mu}r_1) \right], \end{aligned} \quad (4.106)$$

where  $\mu < 0$  and  $c_1, c_2$  are arbitrary constants.

From (4.84) and (4.85) we now need to solve

$$\frac{\partial^2 B}{\partial r^2} - \frac{1}{2r^3} \frac{\partial B}{\partial t} = 0, \quad (4.107)$$

subject to

$$B(r, 0) = r^{1/2} e^{-\mu T/2} \left[ c_1 J_{\sqrt{3}} \left( \sqrt{-\frac{\mu}{r}} \right) + c_2 J_{-\sqrt{3}} \left( \sqrt{-\frac{\mu}{r}} \right) \right], \text{ for } \mu < 0. \quad (4.108)$$

Taking Laplace Transforms of (4.107) with respect to  $\bar{t}$ , we get

$$\frac{d^2 \tilde{B}(r, p)}{dr^2} - \frac{1}{2r^3} [p\tilde{B}(r, p) - B(r, 0)] = 0,$$

which we can write as

$$r^3 \frac{d^2 \tilde{B}}{dr^2} - \frac{p}{2} \tilde{B} = -\frac{B(r, 0)}{2}. \quad (4.109)$$

The solution to (4.109) may be written as  $\tilde{B} = \tilde{B}_c + \tilde{B}_p$  where  $\tilde{B}_c$  is the solution to the corresponding homogeneous equation of (4.109), i.e.

$$r^3 \frac{d^2 \tilde{B}_c}{dr^2} - \frac{p}{2} \tilde{B}_c = 0, \quad (4.110)$$

and  $\tilde{B}_p$  is a particular solution of (4.109). Equation (4.110) has the solution:

$$\tilde{B}_c(r, p) = r^{1/2} \left[ A(p) J_1 \left( -i \sqrt{\frac{2p}{r}} \right) + D(p) Y_1 \left( -i \sqrt{\frac{2p}{r}} \right) \right],$$

where  $A(p)$  and  $D(p)$  are arbitrary functions of the transform variable  $p$  and  $J_\nu$  and  $Y_\nu$  are Bessel functions of the first and second kind respectively.

As an example if we let  $A(p) = c_3 K_1(a\sqrt{p})$  and  $D(p) = c_4 K_1(a\sqrt{p})$ ,  $a = -i$ , where  $K_1$  is the modified Bessel function of order 1 then

$$\tilde{B}_c(r, p) = r^{1/2} \left[ c_3 K_1(a\sqrt{p}) J_1 \left( -\sqrt{\frac{2p}{r}} i \right) + c_4 K_1(a\sqrt{p}) Y_1 \left( -\sqrt{\frac{2p}{r}} i \right) \right],$$

hence

$$B_c(r, \bar{t}) = \mathcal{L}^{-1} \{ \tilde{B}_c(r, p) \}.$$

This formally leads to

$$B_c(r, \bar{t}) = \frac{r^{1/2}}{2\bar{t}} e^{-\frac{[-1 + 2/r]}{4\bar{t}}} \left[ c_3 J_1 \left( -\frac{1}{2\bar{t}} \sqrt{\frac{2}{r}} \right) + c_4 Y_1 \left( -\frac{1}{2\bar{t}} \sqrt{\frac{2}{r}} \right) \right],$$

which may be verified as a solution to (4.107) by direct substitution.

To find  $\tilde{B}_p$ , a particular solution to (4.109) we use the result outlined in (4.96).

Therefore, if in (4.96) we let

$$y_1 = r^{1/2} A(p) J_1 \left( -\sqrt{\frac{2p}{r}} i \right), \quad y_2 = r^{1/2} D(p) Y_1 \left( -\sqrt{\frac{2p}{r}} i \right), \quad \frac{g}{f_2} = -\frac{B(r, 0)}{2r^3},$$

$$W = y_1(y_2)_r - y_2(y_1)_r,$$

where

$$B(r, 0) = r^{1/2} e^{-\mu T/2} \left[ c_1 J_{\sqrt{3}} \left( \sqrt{-\frac{\mu}{r}} \right) + c_2 J_{-\sqrt{3}} \left( \sqrt{-\frac{\mu}{r}} \right) \right] \quad \text{for } \mu < 0,$$

then

$$\begin{aligned} \tilde{B}_p = & \alpha r^{1/2} K_1(-i\sqrt{p}) \left\{ Y_1 \left( -\sqrt{\frac{2p}{r}} i \right) \int \frac{K_1(-i\sqrt{p}) J_1 \left( -\sqrt{\frac{2p}{r}} i \right) h(r) dr}{W} \right. \\ & \left. - J_1 \left( -\sqrt{\frac{2p}{r}} i \right) \int \frac{K_1(-i\sqrt{p}) Y_1 \left( -\sqrt{\frac{2p}{r}} i \right) h(r) dr}{W} \right\} \end{aligned} \quad (4.111)$$

where

$$\begin{aligned} W = & r^{1/2} K_1(-i\sqrt{p}) \left\{ J_1 \left( -\sqrt{\frac{2p}{r}} i \right) \frac{\partial}{\partial r} \left[ r^{1/2} K_1(-i\sqrt{p}) Y_1 \left( -\sqrt{\frac{2p}{r}} i \right) \right] \right. \\ & \left. - Y_1 \left( -\sqrt{\frac{2p}{r}} i \right) \frac{\partial}{\partial r} \left[ r^{1/2} K_1(-i\sqrt{p}) J_1 \left( -\sqrt{\frac{2p}{r}} i \right) \right] \right\}, \text{ and} \end{aligned}$$

$$h(r) = -\frac{1}{2r^{5/2}} e^{-\mu T/2} \left[ c_1 J_{\sqrt{3}} \left( \sqrt{-\frac{\mu}{r}} \right) + c_2 J_{-\sqrt{3}} \left( \sqrt{-\frac{\mu}{r}} \right) \right] \text{ for } \mu < 0.$$

Although it is not possible to find an exact Inverse Laplace Transform of (4.111), it may be done numerically [23]. In any case we have reduced the problem to quadratures. From (4.77) and (4.106) a solution of the BPE (4.61) with the final condition  $V(r, T) = 1$  is

$$V(r, t) = \frac{r^{3/4}}{c(t)B(r, t)} z(r_1, r_2),$$

where

$$r_1 = \frac{1}{r^{1/2}}, \quad r_2 = \frac{t}{2}, \quad z(r_1, r_2) = \beta c(2r_2) e^{-\mu r_2} r_1^{1/2} \left[ c_1 J_{\sqrt{3}}(\sqrt{-\mu} r_1) + c_2 J_{-\sqrt{3}}(\sqrt{-\mu} r_1) \right],$$

$$B(r, \bar{t}) = \frac{r^{1/2}}{2\bar{t}} e^{-\frac{[-1+2/r]}{4\bar{t}}} \left[ c_3 J_1 \left( -\frac{1}{2\bar{t}} \sqrt{\frac{2}{r}} \right) + c_4 Y_1 \left( -\frac{1}{2\bar{t}} \sqrt{\frac{2}{r}} \right) \right] + \mathcal{L}^{-1}\{\tilde{B}_p\},$$

where  $\tilde{B}_p$  is given in (4.111) and  $\bar{t} = T - t$ .

We note that the stochastic differential equation for the risk-neutral spot rate (when  $w(r) = -2r^{3/2}$ ) for this example takes the form

$$dr = 4r^3 b(r, t) dt - 2r^{3/2} dX,$$

where  $b(r, t) = \frac{u - \lambda w}{w^2}$  whose form we can get from the solution of  $B(r, t)$  by the transformation  $b = B^{-1} \frac{\partial B}{\partial r}$ .

## 4.2.2 Solving equation (4.62) by Laplace Transforms

In the previous section we solved equation (4.62), namely

$$\frac{\partial^2 z}{\partial r_1^2} + \frac{\partial z}{\partial r_2} + Q(r_1, r_2)z = 0,$$

where we had

$$Q(r_1, r_2) = -2r + \frac{ww''}{2} - \frac{(w')^2}{4} - 2\frac{c'(t)}{c(t)}, \quad (4.112)$$

by the method of separation of variables. We also needed  $b(r, t)$  to satisfy

$$w^2 \left[ \frac{\partial b}{\partial r} + b^2 \right] + 2 \frac{\partial}{\partial t} \int b(r, t) dr = 0. \quad (4.113)$$

In addition, the bond price  $V(r, t)$  had to satisfy the final condition  $V(r, T) = 1$ ,

so we had

$$V(r, T) = \frac{(w(r))^{1/2}}{c(T)B(r, T)} z(r_1, T/2) = 1,$$

where  $B(r, t)$  was a solution to

$$\frac{\partial^2 B}{\partial r^2} + \frac{2}{w(r)^2} \frac{\partial B}{\partial t} = 0.$$

Hence we required

$$z(r_1, T/2) = (w(r))^{-1/2} B(r, T) c(T).$$

We are now going to solve (4.62) by Laplace Transforms, hence we first make the substitution  $\bar{r}_2 = T/2 - r_2$ , so that (4.62) becomes

$$\frac{\partial^2 z}{\partial r_1^2} - \frac{\partial z}{\partial \bar{r}_2} + Qz = 0, \quad (4.114)$$

which we solve subject to

$$z(r_1, 0) = (w(r))^{-1/2} B(r, T) \quad (4.115)$$

where  $r_1 = \int \frac{1}{w(r)} dr$  and we have let  $c(t) = 1$ .

We now attempt to solve equation (4.62) with  $Q$  as in (4.64) for the same two functions  $w(r)$  as in the previous section.

**Example-1:  $w(r) = r^2$ :** Equation (4.114) with  $Q$  as in (4.64) becomes

$$\frac{\partial^2 z}{\partial r_1^2} - \frac{\partial z}{\partial \bar{r}_2} + \frac{2}{r} z = 0. \quad (4.116)$$

Taking Laplace Transforms of (4.114) with respect to  $\bar{r}_2$ , we get

$$\frac{d^2 \bar{z}(r_1, p)}{dr_1^2} - [p\bar{z} - z(r_1, 0)] + \frac{2}{r_1} \bar{z} = 0, \quad (4.117)$$

$$\frac{d^2 \bar{z}}{dr_1^2} + \bar{z} \left[ \frac{2}{r_1} - p \right] = -z(r_1, 0)$$



where  $\bar{z}(r_1, p) = \mathcal{L}\{z(r_1, \bar{r}_2)\}$ .

Solving the corresponding homogeneous equation of (4.117) we solve

$$\frac{d^2 \bar{z}}{dr_1^2} + \bar{z} \left[ \frac{2}{r_1} - p \right] = 0,$$

whose solution is given as

$$\bar{z}_c(r_1, p) = A(p)M\left(\frac{1}{\sqrt{p}}, \frac{1}{2}, 2\sqrt{p}r_1\right) + D(p)W\left(\frac{1}{\sqrt{p}}, \frac{1}{2}, 2\sqrt{p}r_1\right), \quad (4.118)$$

where  $\bar{z}_c(r_1, p)$  is the complementary solution to (4.117), noting that  $M(\cdot)$  and  $W(\cdot)$  are Whittaker functions (see [1] for definition). To find  $\bar{z}_p$ , the particular solution to (4.117), we use the result as outlined in (4.96). Therefore in (4.96) we let

$$\begin{aligned} y_1 &= A(p)M\left(\frac{1}{\sqrt{p}}, \frac{1}{2}, 2\sqrt{p}r_1\right), \\ y_2 &= D(p)W\left(\frac{1}{\sqrt{p}}, \frac{1}{2}, 2\sqrt{p}r_1\right), \\ \frac{g}{f_2} &= -z(r_1, 0) = r_1 B(-\frac{1}{r_1}, T), \end{aligned}$$

where  $B(r, t)$  is any solution (for  $w = r^2$ ) of

$$\frac{\partial^2 B}{\partial r^2} + \frac{2}{w^2} \frac{\partial B}{\partial t} = 0. \quad (4.119)$$

Hence

$$\bar{z}_p(r_1, p) = y_1 \int y_2 \frac{g}{f_2} \frac{dr}{W} - y_2 \int y_1 \frac{g}{f_2} \frac{dr}{W}, \quad (4.120)$$

where

$$W = y_1(y_2)_r - y_2(y_1)_r.$$

Then

$$z = \mathcal{L}^{-1} \{ \tilde{z}_c(r_1, p) + \tilde{z}_p(r_1, p) \} \quad (4.121)$$

where  $\tilde{z}_c(r_1, p)$  is given by (4.118) and  $\tilde{z}_p(r_1, p)$  is given by (4.120).

We list some solutions of (4.119) in Table (4.3) that we obtained by Laplace Transforms and Separation of Variables and in Table (4.4) that we obtained by Lie Symmetry Analysis [5].

Solutions to (4.119) obtained by taking Laplace Transforms of (4.119)

with respect to  $t$  subject to  $B(r, 0) = 0$ .

$$\frac{r}{2\sqrt{\pi t^{3/2}}} e^{-\frac{1}{4t} \left( a^2 - \frac{2}{r^2} \right)} \left[ \sin \left( \frac{a}{\sqrt{2rt}} \right) \left( a + \frac{\sqrt{2}}{r} \right) + \cos \left( \frac{a}{\sqrt{2rt}} \right) \left( a - \frac{\sqrt{2}}{r} \right) \right]$$

for  $a \geq \frac{\sqrt{2}}{r}$

$$\frac{r}{\sqrt{\pi t}} e^{-\frac{1}{4t} \left( a^2 - \frac{2}{r^2} \right)} \left[ \cos \left( \frac{a}{\sqrt{2rt}} \right) + \sin \left( \frac{a}{\sqrt{2rt}} \right) \right] \text{ for } a \geq \frac{\sqrt{2}}{r}$$

$$\frac{r}{\sqrt{\pi t}} e^{-\frac{1}{4t} \left( a^2 - \frac{2}{r^2} \right)} \left[ \sin \left( \frac{a}{\sqrt{2rt}} \right) \left( \frac{1}{\sqrt{2rt}} + 1 \right) + a \cos \left( \frac{a}{\sqrt{2rt}} \right) \right] \text{ for } a \geq \frac{\sqrt{2}}{r}$$

$$\frac{r}{\sqrt{\pi t}} e^{-\frac{1}{4t} \left( a^2 - \frac{2}{r^2} \right)} \left[ \cos \left( \frac{a}{\sqrt{2rt}} \right) \left( \frac{1}{1 - \sqrt{2rt}} \right) + \frac{a}{2t} \sin \left( \frac{a}{\sqrt{2rt}} \right) \right] \text{ for } a \geq \frac{\sqrt{2}}{r}$$

Note: that in the above solutions we can have  $+r$  added to these (when  $B(r, 0) = r$   $\tilde{B}_p = r/p$ )

Solution to (4.119) obtained by Separation of Variables

$$e^{-\lambda t} \left[ A \sin \left( \frac{\sqrt{2}\sqrt{-\mu}}{r} \right) + B \cos \left( \frac{\sqrt{2}\sqrt{-\mu}}{r} \right) \right] \text{ for } \mu < 0$$

Table 4.3: Solutions to equation (4.119) for  $B(r, t)$  by Laplace Transforms and Separation of Variables

$B(r, t)$ as solutions to (4.119) for $w = r^2$
Solutions obtained by Symmetry Analysis
$\frac{1}{\alpha r t^{1/2} e^{\frac{1}{2r^2 t}}}$ $\alpha_1 r + \alpha_2$ $\left( \frac{\alpha_1 r}{\sqrt{t}} + \frac{\alpha_2}{t^{3/2}} \right) e^{\frac{1}{2r^2 t}}$

Table 4.4: Solutions for  $B(r, t)$  (equation (4.119)) by Symmetry Analysis

Hence from (4.77) a solution of the BPE (4.61) is

$$V(r, t) = \frac{(w(r))^{1/2} z(r_1, r_2)}{B(r, t)}$$

where  $w(r) = r^2$ ,  $B(r, t)$  is any solution of (4.119) as outlined in Table (4.3) and Table (4.4), and  $z(r_1, T/2 - r_2)$  with  $r_1 = -\frac{1}{r}$  and  $r_2 = t/2$  is given in (4.121) with  $\tilde{z}_c$  and  $\tilde{z}_p$  given in (4.118) and (4.120) respectively.

In this case the stochastic differential equation for the risk-neutral spot rate (with  $w(r) = r^2$ ) takes the form

$$dr = r^4 b(r, t) dt + r^2 dX,$$

where  $b(r, t) = \frac{u - \lambda w}{w^2}$  whose form we can get from any of the solutions of  $B(r, t)$  (as outlined in Table (4.3) and Table (4.4)) by the transformation  $b = B^{-1} \frac{\partial B}{\partial r}$ .

**Example-2:  $w = -2r^{3/2}$ :** Equation (4.114) now becomes

$$\frac{\partial^2 z}{\partial r_1^2} - \frac{\partial z}{\partial \bar{r}_2} - \frac{11}{4} \frac{1}{r_1^2} z = 0, \quad (4.122)$$

which we solve subject to

$$z(r_1, 0) = (w(r))^{-1/2} B(r, T). \quad (4.123)$$

Taking the Laplace transform of (4.122) with respect to  $\bar{r}_2$ , we get

$$\frac{d^2 \bar{z}}{dr_1^2} - (p \bar{z}(r_1, p) - z(r_1, 0)) - \frac{11}{4} \frac{1}{r_1^2} \bar{z}(r_1, p) = 0,$$

i.e.,

$$\frac{d^2 \bar{z}}{dr_1^2} - \left( p + \frac{11}{4} \frac{1}{r_1^2} \right) \bar{z}(r_1, p) = -z(r_1, 0). \quad (4.124)$$

Solving the corresponding homogeneous equation of (4.124), namely

$$\frac{d^2 \bar{z}}{dr_1^2} - \left( p + \frac{11}{4} \frac{1}{r_1^2} \right) \bar{z}(r_1, p) = 0,$$

we get

$$\bar{z}_c(r_1, p) = r_1^{\frac{1}{2}} \left( A(p) J_{\sqrt{3}}(\sqrt{p} r_1 i) + D(p) Y_{\sqrt{3}}(\sqrt{p} r_1 i) \right). \quad (4.125)$$

To find  $\bar{z}_p$ , the particular solution to (4.124) we use the result outlined in (4.96).

Therefore in (4.96) we let

$$\begin{aligned} y_1 &= A(p) J_{\sqrt{3}}(\sqrt{p} r_1 i), \\ y_2 &= D(p) Y_{\sqrt{3}}(\sqrt{p} r_1 i), \\ \frac{g}{f_2} &= -z(r_1, 0) = r^{-3/4} B(r, T), \end{aligned}$$

where  $B(r, t)$  can be any solution of

$$\frac{\partial^2 B}{\partial r^2} + \frac{1}{2r^3} \frac{\partial B}{\partial t} = 0 \quad (4.126)$$

such as those found in Table (4.5).

Hence

$$\tilde{z}_p(r_1, p) = y_1 \int y_2 \frac{g}{f_2} \frac{dr}{W} - y_2 \int y_1 \frac{g}{f_2} \frac{dr}{W}, \quad (4.127)$$

where

$$W = y_1(y_2)_r - y_2(y_1)_r.$$

Then

$$z = \mathcal{L}^{-1} \{ \tilde{z}_c(r_1, p) + \tilde{z}_p(r_1, p) \} \quad (4.128)$$

where  $\tilde{z}_c$  is given by (4.125) and  $\tilde{z}_p$  is given by (4.127).

$B(r, t)$ as solutions to (4.126) for $w = -2r^{3/2}$
By Symmetry Analysis
$\frac{1}{c_1 r e^{\frac{1}{2rt}}} + \frac{c_2}{t^2} e^{\frac{1}{2rt}}$ $Q(rt) \text{ where } Q(z) = \alpha \int \frac{1}{e^{\frac{1}{2z}}} dz + \beta$
By Separation of Variables
$e^{-\mu t} \left[ A r^{1/2} J_1 \left( \frac{\sqrt{2} \sqrt{-\mu}}{\sqrt{r}} \right) + B r^{1/2} Y_1 \left( \frac{\sqrt{2} \sqrt{-\mu}}{\sqrt{r}} \right) \right], \text{ for } \mu < 0$
By Laplace Transforms with respect to $t$
$\frac{r^{1/2}}{2t} e^{-\frac{(a^2 - 2/r)}{4t}} \left[ J_1 \left( \frac{-a}{\sqrt{2rt}} \right) + Y_1 \left( \frac{-a}{\sqrt{2rt}} \right) \right], \text{ for } a \geq -\frac{\sqrt{2}}{r}$

Table 4.5: Solutions for  $B(r, t)$  (equation (4.126))

From (4.77), a solution of the BPE (4.61) is

$$V(r, t) = \frac{r^{3/4} z(r_1, r_2)}{B(r, t)}$$

where  $B(r, t)$  is any solution of (4.126) as outlined in Table (4.5), and  $z(r_1, T/2 - r_2)$  with  $r_1 = \frac{1}{r^{1/2}}$  and  $r_2 = \frac{t}{2}$  is given in (4.128) with  $\tilde{z}_c$  and  $\tilde{z}_p$  given in (4.125) and (4.127) respectively.

In this case the stochastic differential equation for the risk-neutral spot rate (when  $w(r) = -2r^{3/2}$ ) takes the form

$$dr = 4r^3 b(r, t) dt - 2r^{3/2} dX,$$

where  $b(r, t) = \frac{u - \lambda w}{w^2}$  whose form we can get from any of the solutions of  $B(r, t)$  (as outlined in Table (4.5)) by the transformation  $b = B^{-1} \frac{\partial B}{\partial r}$ .

### 4.2.3 Reducing equation (4.62) to the Heat Equation

In section (4.1) we reduced the Bond Pricing Equation

$$\frac{\partial^2 V}{\partial r^2} + 2b(r, t) \frac{\partial V}{\partial r} + \frac{2}{w^2} \frac{\partial V}{\partial t} - \frac{2r}{w^2} V = 0,$$

where

$$b(r, t) = \left( \frac{u - \lambda w}{w^2} \right)$$

to

$$\frac{\partial^2 z}{\partial r_1^2} + \frac{\partial z}{\partial r_2} + Q(r_1, r_2)z = 0,$$

where  $Q(r_1, r_2)$  was given by

$$\begin{aligned} Q(r_1, r_2) = & -mr - \frac{mww'}{2} \left( b(r, t) - \frac{w'}{2w} \right) \\ & - \frac{w^2 m}{2} \left[ \frac{\partial b}{\partial r} - \frac{1}{2} \left( \frac{ww'' - (w')^2}{w^2} \right) + \left( b(r, t) - \frac{w'}{2w} \right)^2 \right] \\ & - m \left( \frac{c'(t)}{c(t)} + \frac{\partial}{\partial t} \int b(r, t) dr \right). \end{aligned}$$

and

$$\begin{aligned} t &= mr_2 + a_1 \\ \int \frac{1}{w(r)} dr &= \sqrt{\frac{m}{2}} r_1 + a_2 \\ z &= H(r, t)V, \text{ where} \\ H(r, t) &= w^{-1/2} c(t) \exp \int b(r, t) dr \end{aligned}$$

where  $a_1$ ,  $a_2$  and  $m$  were arbitrary constants and  $c(t)$  was an arbitrary function of  $t$ . In Section 4.2 we let  $m = 2$  and chose  $Q(r_1, r_2)$  to be of the form

$$\begin{aligned} Q(r_1, r_2) &= F(r_1) + G(r_2) \\ &= -2r + \frac{ww''}{2} - \frac{(w')^2}{4} - 2 \frac{c'(t)}{c(t)} - \zeta \end{aligned}$$

so that  $b(r, t)$  needed to satisfy the equation



$$w^2 \left( \frac{\partial b}{\partial r} + b^2 \right) + 2 \frac{\partial}{\partial t} \int b(r, t) dr = \zeta \quad (4.129)$$

where  $\zeta$  could be  $\zeta(r)$ ,  $\zeta(t)$ ,  $\zeta_1(r) + \zeta_2(t)$  or a constant. We solved the above problem with  $\zeta = 0$ .

However if we now choose  $\zeta = \zeta(r)$  so that

$$\zeta(r) = -2r + \frac{ww''}{2} - \frac{(w')^2}{4} \quad (4.130)$$

and  $c'(t) = 0$ , then equation (4.62) becomes the backward constant-coefficient heat equation

$$\frac{\partial^2 z}{\partial r_1^2} + \frac{\partial z}{\partial r_2} = 0, \quad (4.131)$$

for which many solutions are known and tabulated [6].

We solve (4.131) by the method of Laplace Transforms which will incorporate the final condition for the solution. Without loss of generality we let  $c(t) = 1$ .

We recall then that

$$V(r, t) = \frac{(w(r))^{1/2} z(r_1, r_2)}{B(r, t)},$$

where  $b = B_r/B$  satisfies (4.129). Performing a similar calculation to that in Section 4.2, we find that  $B$  needs to satisfy

$$\frac{w^2}{2}B_{rr} + B_t = \frac{\zeta(r)}{2}B. \quad (4.132)$$

The requirement that  $V(r, T) = 1$  implies that

$$z(r_1, T/2) = (w(r))^{-1/2}B(r, T).$$

To solve equation (4.131) by Laplace Transforms we first do a change of variable by letting  $\bar{r}_2 = T/2 - r_2$  giving equation (4.131) as

$$\frac{\partial^2 z}{\partial r_1^2} - \frac{\partial z}{\partial \bar{r}_2} = 0, \quad (4.133)$$

which we solve subject to

$$z(r_1, 0) = (w(r))^{-1/2}B(r, T). \quad (4.134)$$

Taking Laplace Transforms of (4.133) with respect to  $\bar{r}_2$  we get

$$\frac{\partial^2 \tilde{z}}{\partial r_1^2} - p\tilde{z} = -z(r_1, 0). \quad (4.135)$$

Solving the corresponding homogeneous equation of (4.135) for  $p > 0$  we get

$$\tilde{z}_c = A(p)e^{\sqrt{p}r_1} + D(p)e^{-\sqrt{p}r_1} \quad (4.136)$$

where  $\tilde{z}_c$  denotes the complementary solution to (4.135). There are many possible choices for the arbitrary functions  $A(p), D(p)$  of the transform variable as we saw in Table (4.2). In fact all the choices that are listed in Table (4.2) are appropriate and hence give us a choice of 324 different solutions.

As an example if  $A(p) = 0$  and  $D(p) = c_2 + c_1 p^{-1/2}$  we get after inverting (4.136)

$$z_c(r_1, \bar{r}_2) = \frac{c_1}{(\pi \bar{r}_2)^{1/2}} \exp\left(\frac{-r_1^2}{4\bar{r}_2}\right) - \frac{c_2 r_1}{\sqrt{\pi \bar{r}_2^3}} \exp\left(\frac{-r_1^2}{4\bar{r}_2}\right) \quad (4.137)$$

where  $\bar{r}_2 = \frac{T}{2} - r_2$ .

To find  $\tilde{z}_p$ , the particular solution to (4.135), we use the result (4.96). For now we leave  $z(r_1, 0)$  general.

Letting

$$y_1 = A(p)e^{\sqrt{p}r_1} \quad \text{and} \quad y_2 = D(p)e^{-\sqrt{p}r_1}$$

then the Wronskian  $W = y_1(y_2)_r - y_2(y_1)_r$  simplifies to

$$W = 2\sqrt{p}A(p)D(p)$$

and

$$\frac{g}{f_2} = -z(r_1, 0).$$

We then substitute this into

$$\tilde{z}_p = y_2 \int y_1 \frac{g}{f_2} \frac{dr_1}{W} - y_1 \int y_2 \frac{g}{f_2} \frac{dr_1}{W} \quad (4.138)$$

to find the particular solution to (4.135).

**Example-1:**  $\mathbf{w}(\mathbf{r}) = \mathbf{r}^2$ : After substituting into (4.138) we get with a little simplification,

$$\tilde{z}_p = \frac{e^{\frac{\sqrt{p}}{r}}}{\sqrt{p}} \int_0^r e^{-\frac{\sqrt{p}}{r_3}} H(r_3) dr_3 - \frac{e^{-\frac{\sqrt{p}}{r}}}{\sqrt{p}} \int_0^r e^{\frac{\sqrt{p}}{r_3}} H(r_3) dr_3 \quad (4.139)$$

where  $H(r) = \frac{z(r_1, 0)}{2r^2}$  (recalling that  $r_1 = -\frac{1}{r}$ ). We now need to find  $z_p = \mathcal{L}^{-1}(\tilde{z}_p)$ .

If we let

$$\begin{aligned} z_e(r_1, \bar{r}_2) &= \mathcal{L}^{-1} \left\{ \frac{e^{\frac{\sqrt{p}}{r}}}{\sqrt{p}} \int_0^r e^{-\frac{\sqrt{p}}{r_3}} H_e(r_3) dr_3 \right\}, \text{ and} \\ z_o(r_1, \bar{r}_2) &= \mathcal{L}^{-1} \left\{ \frac{e^{\frac{\sqrt{p}}{r}}}{\sqrt{p}} \int_0^r e^{-\frac{\sqrt{p}}{r_3}} H_o(r_3) dr_3 \right\} \end{aligned} \quad (4.140)$$

where

$$\begin{aligned} H_e(r) &= \frac{H(r) + H(-r)}{2}, \text{ the even part of } H \text{ and} \\ H_o(r) &= \frac{H(r) - H(-r)}{2}, \text{ the odd part of } H \end{aligned}$$

then

$$z_p(r_1, \bar{r}_2) = z_e(r_1, \bar{r}_2) + z_o(r_1, \bar{r}_2) + z_e(-r_1, \bar{r}_2) - z_o(-r_1, \bar{r}_2) \quad (4.141)$$

We note firstly that

$$\begin{aligned}
\mathcal{L}^{-1} \left\{ e^{\frac{p}{r}} \int_0^r e^{-\frac{p}{r_3}} H_e(r_3) dr_3 \right\} &= \int_0^r \mathcal{L}^{-1} \left( e^{-p(\frac{1}{r_3} - \frac{1}{r})} H_e(r_3) \right) dr_3 \\
&= \int_0^r \delta(\bar{r}_2 - (\frac{1}{r_3} - \frac{1}{r})) H_e(r_3) dr_3.
\end{aligned}$$

With the substitutions  $r_4 = -\frac{1}{r_3}$  and  $r_1 = -\frac{1}{r}$ , we get

$$\int_{-\infty}^{r_1} \delta(\bar{r}_2 + r_4 - r_1) H_e(-\frac{1}{r_4}) \frac{1}{r_4^2} dr_4$$

which evaluates to

$$H_e \left( [\bar{r}_2 - r_1]^{-1} \right) \left( \frac{1}{\bar{r}_2 - r_1} \right)^2.$$

Hence,

$$\begin{aligned}
z_e(r_1, \bar{r}_2) &= \mathcal{L}^{-1} \left\{ \frac{e^{\frac{\sqrt{p}}{r}}}{\sqrt{p}} \int_0^r e^{-\frac{\sqrt{p}}{r_3}} H_e(r_3) dr_3 \right\} \\
&= \frac{1}{\sqrt{\pi \bar{r}_2}} \int_0^\infty e^{-\frac{u^2}{4\bar{r}_2}} \mathcal{L}^{-1} \left[ e^{\frac{p}{r}} \int_0^r e^{-\frac{p}{r_3}} H_e(r_3) dr_3 \right] du \\
&= \frac{1}{\sqrt{\pi \bar{r}_2}} \int_0^\infty e^{-\frac{u^2}{4\bar{r}_2}} H_e \left( [u - r_1]^{-1} \right) \left( \frac{1}{u - r_1} \right)^2 du.
\end{aligned}$$

Similarly it can be shown that

$$z_o(r_1, \bar{r}_2) = \frac{1}{\sqrt{\pi \bar{r}_2}} \int_0^\infty e^{-\frac{u^2}{4\bar{r}_2}} H_o \left( [u - r_1]^{-1} \right) \left( \frac{1}{u - r_1} \right)^2 du.$$

So we have from (4.139) that

$$\begin{aligned}
z_p(r_1, \bar{r}_2) &= \frac{1}{\sqrt{\pi \bar{r}_2}} \int_0^\infty e^{-\frac{u^2}{4\bar{r}_2}} H\left([u - r_1]^{-1}\right) \left(\frac{1}{u - r_1}\right)^2 du \\
&+ \frac{1}{\sqrt{\pi \bar{r}_2}} \int_0^\infty e^{-\frac{u^2}{4\bar{r}_2}} H\left([u + r_1]^{-1}\right) \left(\frac{1}{u + r_1}\right)^2 du \\
&- \frac{2}{\sqrt{\pi \bar{r}_2}} \int_0^\infty e^{-\frac{u^2}{4\bar{r}_2}} H_o\left([u + r_1]^{-1}\right) \left(\frac{1}{u + r_1}\right)^2 du.
\end{aligned} \tag{4.142}$$

Finally we have

$$z(r_1, \bar{r}_2) = z_c(r_1, \bar{r}_2) + z_p(r_1, \bar{r}_2) \tag{4.143}$$

where an example of  $z_c$  is given in (4.137) and  $z_p$  is given in (4.142).

We now need to look for suitable functions  $B(r, t)$ . From (4.130) and (4.132)

$B(r, t)$  needs to satisfy

$$\frac{w^2}{2} \frac{\partial^2 B}{\partial r^2} + \frac{\partial B}{\partial t} = -rB \tag{4.144}$$

which we solve by the method of separation of variables. Letting  $B(r, t) = X(r)T(t)$

and substituting into (4.144) we find

$$\frac{w^2}{2} \frac{X''}{X} + r = -\frac{T'}{T} = \mu$$

where  $\mu$  is a constant.

The solution to

$$\frac{T'}{T} = -\mu$$

is

$$T(t) = ke^{-\mu t} \quad (4.145)$$

where  $k$  is an arbitrary constant and with  $w(r) = r^2$  the corresponding  $X$  equation becomes

$$X'' + \frac{2(r - \mu)}{r^4} X = 0. \quad (4.146)$$

The solution to (4.146) is

$$X(r) = e^{-\frac{\sqrt{2\mu}}{r}} r \left[ c_3 \Phi \left( -\frac{1}{\sqrt{2\mu}}, 0, \frac{2\sqrt{2\mu}}{r} \right) + c_4 \Psi \left( -\frac{1}{\sqrt{2\mu}}, 0, \frac{2\sqrt{2\mu}}{r} \right) \right],$$

where  $\Phi$  and  $\Psi$  are the Confluent Hypergeometric functions KummerM and KummerU respectively.

Therefore the solution to equation (4.144) is

$$\begin{aligned} B &= X(r)T(t) \\ &= re^{-\mu t} e^{-\frac{\sqrt{2\mu}}{r}} \left[ c_3 \Phi \left( -\frac{1}{\sqrt{2\mu}}, 0, \frac{2\sqrt{2\mu}}{r} \right) + c_4 \Psi \left( -\frac{1}{\sqrt{2\mu}}, 0, \frac{2\sqrt{2\mu}}{r} \right) \right] \end{aligned} \quad (4.147)$$

Hence a solution to the BPE which satisfies the final condition is given as

$$V(r, t) = \frac{(w(r))^{1/2} z(r_1, r_2)}{B(r, t)}$$

where  $B(r, t)$  is given by (4.147),  $w(r) = r^2$ , and  $z(r_1, r_2)$  is given by (4.143).

The stochastic differential equation for the risk-neutral risk rate (when  $w(r) = r^2$ ) takes the form

$$dr = r^4 b(r, t) dt + r^2 dX,$$

where  $b(r, t) = \frac{u - \lambda w}{w^2}$  whose form we can get from equation (4.147) by the transformation  $b = B^{-1} \frac{\partial B}{\partial r}$ .

**Example-2:**  $w(r) = -2r^{3/2}$ : In order to find  $z_p$  for this case we perform a similar calculation to that in the previous example. Noting that now  $r_1 = \frac{1}{\sqrt{r}} > 0$ , we find that in general for  $r_1 > 0$ ,

$$\begin{aligned} z_p(r_1, \bar{r}_2) = & - \frac{1}{\sqrt{\pi \bar{r}_2}} \int_0^\infty e^{-\frac{u^2}{4\bar{r}_2}} H(r_1 - u) du \\ & - \frac{1}{\sqrt{\pi \bar{r}_2}} \int_0^\infty e^{-\frac{u^2}{4\bar{r}_2}} H(-r_1 - u) du \\ & + \frac{2}{\sqrt{\pi \bar{r}_2}} \int_0^\infty e^{-\frac{u^2}{4\bar{r}_2}} H_o(-r_1 - u) du \end{aligned} \quad (4.148)$$

where notation is as in the previous example with  $H(r_1) = \frac{z(r_1, 0)}{2}$ . Then

$$z(r_1, \bar{r}_2) = z_c(r_1, \bar{r}_2) + z_p(r_1, \bar{r}_2) \quad (4.149)$$

where  $z_c$  is given in (4.137) and  $z_p$  is given in (4.148). We now need to look for suitable functions for  $B(r, t)$ .

From (4.130) and (4.132),  $B(r, t)$  needs to satisfy

$$\frac{w^2}{2} \frac{\partial^2 B}{\partial r^2} + \frac{\partial B}{\partial t} = -\frac{11}{8} r B \quad (4.150)$$



which we solve by separation of variables. Letting  $B(r, t) = X(r)T(t)$  and substituting into (4.150) we get

$$\frac{w^2}{2} \frac{X''}{X} + \frac{11}{8} r = -\frac{T'}{T} = \mu$$

where  $\mu$  is an arbitrary constant.

The solution to

$$\frac{T'}{T} = -\mu$$

is

$$T(t) = ke^{-\mu t} \quad (4.151)$$

where  $k$  is an arbitrary constant, and with  $w(r) = -2r^{3/2}$  the corresponding  $X$  equation becomes

$$X'' + \frac{\left(\frac{11}{8r} - \mu\right)}{2r^3} X = 0.$$

which has as its solution

$$X(r) = r^{1/2} \left[ c_3 J_{\frac{\sqrt{15}}{2}} \left( -\sqrt{\frac{-2\mu}{r}} \right) + c_4 J_{-\frac{\sqrt{15}}{2}} \left( -\sqrt{\frac{-2\mu}{r}} \right) \right]. \quad (4.152)$$

Therefore the solution to equation (4.150) is

$$B(r, t) = e^{-\mu t} r^{1/2} \left[ c_3 J_{\frac{\sqrt{15}}{2}} \left( -\sqrt{\frac{-2\mu}{r}} \right) + c_4 J_{-\frac{\sqrt{15}}{2}} \left( -\sqrt{\frac{-2\mu}{r}} \right) \right]. \quad (4.153)$$

Hence a solution to the BPE is

$$V(r, t) = \frac{r^{3/4} z(r_1, r_2)}{B(r, t)}$$

where the solution for  $z(r_1, r_2)$  is as outlined in (4.149) where the initial condition  $z(r_1, 0)$  depends on the solution of  $B(r, t)$  which is given by (4.153).

The stochastic differential equation for the risk-neutral spot rate (when  $w(r) = -2r^{3/2}$ ) takes the form

$$dr = 4r^3 b(r, t) dt - 2r^{3/2} dX,$$

where  $b(r, t) = \frac{u - \lambda w}{w^2}$  whose form we can get from equation (4.153) by the transformation  $b = B^{-1} \frac{\partial B}{\partial r}$ .

In this chapter we constructed a variety of new analytic solutions to the BPE. In the next chapter we conclude with an outlook of the results obtained and the potential for future research work.

# Chapter 5

## Outlook

There are no undisputed economic guidelines associated with modelling interest rates except that they should remain positive (which was defied by Switzerland in the 1960s) and that they should not exhibit exponential growth. Unfortunately that does not give us a very clear focus for modelling purposes.

Even though we have only dealt with a single source of randomness in our model for interest rates, our objective was to get new analytic solutions to the BPE. We first transformed the BPE to a standard canonical form in which only one model-dependent adjustable coefficient function is left. We then found special cases for this coefficient function that allowed either a full general solution of the BPE or the reduction to a single inverse Laplace transform. In some cases, this coefficient function is identically zero, leaving nothing more to solve than the classical heat diffusion equation. We have shown that the forms of the coefficients in the BPE do not have to be restricted to those outlined in Chapter 3, in order to get analytic solutions to

the Bond Pricing Equation. The new solutions that we present in Chapter 4 satisfy the required final condition namely that the solutions should uniformly reach the expiry value 1, independent of  $r$ .

One of the objectives in having these solutions is that we can use them to build yield curves. Some yield curves associated with our solutions were plotted. They exhibited all the 3 typical yield curves observed in the market. We also note that the popular models use the form of  $w$  as  $w = \alpha r^\beta$  where the choice of  $\beta$  is either 0 or 0.5. This was previously done so that closed form solutions to the BPE could be easily obtained. There have been a lot of empirical studies in trying to obtain an estimate for the coefficient  $\beta$  from data. Chan, Karolyi, Longstaff & Sanders (1992) [7] using the Generalised Method of Moments, found that models with  $\beta \geq 1$  capture the dynamics of the short rate better than those with  $\beta < 1$ . We obtained solutions with  $\beta = 1.5$  and with  $\beta = 2$ , which is an extension of previous models.

Our construction of the solutions to the BPE differed significantly from all previous methods. We hope that the solution method presented in Chapter 5 will give us more flexibility when we attempt to solve more complicated systems in the world of Interest Rate Derivatives.

In the end it would be fair to say that if we have only one source of randomness, the spot interest rate, then determining the behaviour of the whole yield curve *can* give unrealistic results. For example the one-factor theory says that that we can hedge a ten-year instrument with a one-year bond, which is clearly not true. To better model real markets one needs to introduce two or more factors. Now there

will be at least two sources of randomness which may allow greater flexibility to fit the observed data.

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